A Strategic Model of Network Formation with Endogenous Link Strength

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Abstract

This paper analyzes formation of networks when players choose how much time to invest in other players. As opposed to the distance-based utility weighted link formation game by Bloch and Dutta (2009) in which only the most reliable path is considered, this model assumes the information can be transferred using all possible paths in the network. We study the model under two different link strength functions. First, we assume the link strength is the arithmetic mean of agents' investment levels, i.e., the investments are perfect substitutes. This specification allows players to form links unilaterally to other players. Second, we assume the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other. We show that, when the investments are perfect substitutes, every player is connected to another either directly or indirectly with no more than two links under any Nash equilibrium. Moreover, we find that the strict Nash equilibrium structure is a star network. On the other hand, using the Cobb-Douglas link strength function, we show that paired networks in which players are matched in pairs, are Nash equilibria. However, we also consider a sequential game in which players choose and announce their investments publicly according to a random ordering. We show that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, we conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium. Lastly, we find that, for both link strength functions, Nash equilibria may not be strongly efficient.

JEL Codes: D85, C72

Keywords: network formation, communication networks, endogenous link strength

1 Introduction

Humans are social creatures; social interactions influence our thinking and behavior. Social networks that we belong to influence our opinions and decisions, determining the products we buy, political candidates for whom we vote, the level of education we obtain, and whether our kids get vaccinations. They also play a central role in the transmission of information such as job opportunities and availability of new technologies. To build better models of human behavior, economists cannot ignore the role of social and economic networks. Hence, following in the long tradition of sociology literature, research on networks in economics has grown rapidly over the last two decades.

The structure of a network, i.e., existence of key players, whether everybody is connected or whether groups are segregated, affects the diffusion of information; thus, it's important to gain insight into the network structures likely to emerge and how these structures are related to the ones optimal for a society. Following the seminal papers of Jackson and Wolinsky (1996), and Bala and Goyal (2000), the majority of the literature dealing with models of network formation assumes that the agents make binary decisions; they choose whether or not to link to another agent. However, in a wide variety of situations, such as friendships, sharing of information, and trade of goods and services, agents decide not only whom to connect to but also how much time to spend on each connection they make. Even though treating links as binary quantities helps to overcome computational difficulties, this simplification doesn't allow for analysis of many applications with differing link intensity.

The first work to point out the importance of allowing for richer environments for links strengths in network analysis is that of Granovetter (1973). Granovetter interviewed a hundred people and sent out two hundred questionnaires in the Boston area in the late 1960s to analyze how people find their jobs. The results show that more than half of the people found their jobs through personal contacts. However, the surprising result of the study is that, in most cases, the information on job opportunities came from the people who are not close to the job seeker. Granovetter argues that weak ties play a key role in information transmission within networks because such ties connect people who are dissimilar, and therefore, have nonoverlapping groups of friends.

In this paper, we analyze the formation of networks when players choose how much time to invest in other players. Our analysis is centered around information and friendship networks in which players invest in relationships to exchange information or favours. However, our model is applicable to any situation where players exchange divisible goods. Specifically, we analyze a network formation game in which each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. The link strength is a function of investments of the players involved in the link. Once a direct link is formed, the information is transferred both ways with decay. Therefore, an agent's investment decision about a link not only affects his direct benefit from the relationship but also that of the other agent involved in the link. Moreover, we assume that there are benefits from indirect communication.

The extent to which link externalities are accounted for has been limited in the previous literature following the seminal papers by Jackson and Wolinsky (1996), and Bala and Goyal (2000). While calculating the indirect benefits, models using distance-based utility, such as Jackson and Wolinsky's connections model and Bala and Goyal's two-way benefit model, only consider the shortest path between the agents. There is no value added in having multiple paths between agents. However, in wide variety of situations, when the good exchanged in the network is divisible, like information, the good is transferred by using all the paths between the agents. Therefore, our model assumes the information can be transferred using all possible paths in the network. However, in order to add tractability to our analysis, following Brueckner (2006), we assume that the benefits from indirect information transfers are zero when two agents are connected by more than two links. That is, when two agents, i and j, are directly connected, agent i can only obtain indirect information of agent j's direct links via agent j.

Heterogeneity between agents are allowed in terms of their intrinsic value. We assume that individuals are ranked according to their intrinsic value of information. Therefore, the value of a direct link for agent i with agent j depends on three variables: intrinsic value of agent j, the value of the other agents whom agent j is directly connected, and the level of investment of agent j.

One of the challenges of modeling network formation with endogenous link strengths is transforming investments into the link strengths. We study the model under two different link strength functions to model different situations. First, we assume that link strength is the arithmetic mean of agents' investment levels, that is, the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with other players. Therefore, it is reminiscent of the model by Bala and Goyal (2000). Alternatively, in the second case, we assume that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other, which is similar to Jackson and Wolinsky's (1996) model.

In order to motivate two different types of link strength functions, we introduce several applications of decentralized information and innovation diffusion systems. The main distinction between the different specifications of the link strength function is the availability of the information and the element of consent to obtain intrinsic information. Additively separable link strength function is more appropriate for the situations when intrinsic information of an agent is available publicly; however, other agents need to invest time in learn about this information. Whereas, Cobb-Douglas is applicable to the situations in which both agents are required to invest in a relationship in order to exchange information. Even though many applications have elements of both, we provide some applications that correspond closely to one compared to the other.

Legitech is one of the examples of a decentralized system of innovation diffusion discussed in E. M. Rogers (1983). Legitech is a computer conferencing system used for exchanging information among legislative staffs of various states. It works like an internet forum: a legislator who is seeking advice can send out a general inquiry on the topic to solicit suggestions over the Legitech computer network to find how other states have responded to this problem. Legislators in other states can respond to the inquiry. The responses can be a specific technical solution or a reference to resources that can supply an answer, such as a reference to a bill originated by a legislator in another state. Other members of the network also have access to answers to others' requests. In terms of the network structure observed, Rogers (1983) notes that certain information sources in Legitech have gained reputation and respect of others on the system for their careful and competent responses to inquiries. Thus, legislators are more likely to follow their advice. In this system, the legislator who is seeking advice mainly bears the cost of obtaining information, since the cost of posting a response, especially the ones consisting of a reference to a source, is low; while the legislator who is seeking advice have to invest in studying the posted reference. Therefore, additively separable link strength function is more appropriate in characterizing this system.

Another application of additively separable link strength function presented in Rogers (1983) is the use of models and on-the-spot conferences in communities for diffusion of innovation regarding health, family planning, and industrial development. A model is defined as a local unit that pioneers in inventing and developing an innovation, in evaluating its results, and in serving as an example for the diffusion of the innovation to other units. The models often have exemplary characteristics of success so that many people want to learn about their techniques. In order to facilitate information transmission, on-the-spot conferences are held at the site of a model. During these meetings, participants observe the innovation in use by a local unit and are able to ask questions about the implementation of the innovation and its effectiveness. The participants, then, decide whether or not to adopt the innovation, and, if they decide to adopt, how to incorporate it into their particular local conditions. Once a participant decides to adopt a technology, the participant not only uses the knowledge of the model but also contributes to it with his experience of the technology. Thus, the flow of benefits is two-way even though only the participant invests in the relationship. Rogers (1983) states that the innovation demonstrated at the exemplary model need not be copied exactly. It is observed that often a great degree of variety can be observed in the forms of an innovation that are actually implemented by local units.

Rogers' (1983) conclusions are also in accordance with the findings of Conley and Udry (2010). They investigate the role of social learning in the diffusion of a new agricultural technology in Ghana using data on farmers' communication patterns. They find that farmers align their inputs according to the information received from the neighbors who were surprisingly successful in previous periods.

In these three applications, the general theme is the emergence of pioneer agents, such as the top contributors in Legitech and the most successful farmers in Ghana. Because of their prestige and success, these pioneer agents gain popularity and influence others' innovation decisions. We show that strict Nash equilibrium of the model using additively separable function is a star network in which there exists a center player such that all other players are connected to. Hence, our results are in accordance with the emergence of the pioneer agents.

Lastly, our motivation for Cobb-Douglas link strength function comes from the book by Blau (1963). Blau (1963) studies consultation patterns among agents working in a federal law enforcement agency. In this law enforcement agency, agents are responsible for the inspection of business establishments and preparing reports on the firms' compliance with the law. As the tasks include complex legal regulations and the reports might lead to legal action against the firms, the agents often need consultation with the other agents. Blau (1963) points out that a consultation is an exchange of values in which both participants gains something by paying a price. The agent seeking advice is able to perform better than he could without receiving any help. By asking for an advice, however, he not only has to spend time explaining his problem to his colleague but also implicitly acknowledge his incompetency to solve a problem. On the other hand, while the consultant has to devote his time to the consultation and disrupt his own work, he gains prestige in return. The final pattern of this social structure is different than what he expects to get: instead of asking advice from a highly competent agent, agents establish partnerships of mutual consultation and less competent agents tend to pair off as partners[1]. In our paper, using the Cobb-Douglas link

[1] Latest civil service rating by supervisor is used for competency measure.

strength function, we show that paired networks in which players are matched in pairs, are Nash equilibria. However, when we consider a sequential game in which players choose and announce their strategies publicly according to a random ordering, we find that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the only subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, we conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium. Hence, our results are analogous of Blau (1963)'s final pattern, in which the agents are matched in pairs according to their competency levels.

The rest of the paper proceeds as follows. The next section, Section 2, presents relevant economics literature on network formation games. Section 3 formally introduces the general framework and proves the existence of Nash equilibrium for any continuous, non-decreasing concave link strength function. Section 4 analyzes the case with additively separable link strength function, while Section 5 considers the case with Cobb-Douglas link strength function. Finally, Section 6 concludes. A full characterization of Nash equilibrium and surplus-maximizing outcome for three player game with additively separable link strength is included in Appendix A.

2 Relevant Literature

We start this section with the literature on network formation games with binary link strengths, before introducing the works on weighted link formation games. The literature on network formation games starts with a simultaneous-move game by Myerson (1977), introduced in the context of the formation of communication graphs. In this model, each player simultaneously announces the set of players with whom he would like to be linked. The links are formed if both players involved in the link named each other. However, the main weakness of this model is that it has too many Nash equilibria, including complete network, in which every player is directly linked to other players, and empty network, in which no links are formed. Therefore, this analysis fails to capture the idea that two individuals should communicate and engage in relationship if it is in their mutual benefit.

Later, Aumann and Myerson (1988) propose an extensive form game based on an ordering over all possible links. This model provides the pairs of individuals with the opportunity to communicate and reconsider their decision if the link is not formed in the previous rounds. When a link appears in the ordering, the pair of players involving that link decide on whether or not to form that link knowing the decisions of all pairs coming before them. A decision to form a link is binding and cannot be undone. The game moves through all links initially. If at least one link is formed during the first round, then it starts from the same ordering of links again; however, it moves through only the links that are not formed. Therefore, if a pair of players (i, j) decide not to form a link initially, but some other pair coming after them forms a link, then the pair (i, j) is allowed to reconsider their decision. The game continues until either all links are formed, or there is a round that no new links have formed even though the links that haven't been formed have been reconsidered. Since this is a finite game with perfect information, it always has a subgame perfect equilibrium. However, even in simple settings, it can be very difficult to solve the game by using backward induction. Another shortcoming is that the ordering of the links can have a serious impact on the network structures at the equilibrium.

In Myerson (1977) and Aumann and Myerson (1988), the standard game-theoretic analysis fails to account for the communication and coordination between the agents properly, and provide an insight on why and how the structures at the equilibrium are formed. In order to overcome these issues, Jackson and Wolinsky (1996) introduce a new concept of stability: pairwise stability. A network is defined to be pairwise stable if no pairs of unlinked players both want to form a link, and no player wants to break off a link. This notion of stability is based upon the idea that two players should be able to form a link if it is mutually beneficial to do so. Therefore, formation of a link should involve mutual consent of the individuals to be linked. Jackson and Wolinsky (1996), then, use this new concept of equilibrium to solve two models, connections and co-author models, introduced in the same paper.

The connections model is a simple model of social connections in which links represent social relationships. These relationships offer benefits such as information and favors. However, it is costly to form links. Players directly communicate with those to whom they are linked. There are also benefits from indirect communication from those to whom their adjacent links are connected. The value of the benefit from the indirect links decays with the distance of the relationship. When calculating the indirect benefits, if there are multiple paths between the players, only the shortest path is considered. Jackson and Wolinsky (1996) analyze the model under the assumption that the agents are symmetric, i.e., the cost and the value of a link is the same for all players. They find that a pairwise stable network has at most one non-empty component. They show that a complete network is the unique pairwise stable network in the low cost range; a star, in which there exists a player (the center) such that all other players are connected to, is pairwise stable but not necessarily the unique one in the medium cost range; and each player has either no links or at least two links in any pairwise stable network in the high cost range. Moreover, they characterize strongly efficient networks, the network structures that maximize the total utility of the agents. They show that the unique strongly efficient network is the complete network in the low cost range, a star in the medium cost range, and empty network in the high cost range. In addition, they show the star network is efficient but not pairwise stable for a wide range of parameters. [2]

Furthermore, in the co-author model, players are interpreted as researchers who spend time writing papers. Each link represents a collaboration between a pair of researchers. Each collaboration creates a synergy depending on the time how much they spend together. Moreover, since each player has a fixed amount of time spend on research, the amount of time a player spends on a collaboration decreases with the number of links that the player has. Hence, contrary to connections model in which the indirect communication has benefits, in the co-author model, indirect connections create distractions and result in negative externalities. Jackson and Wolinsky (1996) show that the network consisting of pairs are the strongly efficient networks; and pairwise stable networks can be partitioned into fully intraconnected components, and tend to be over-connected from an efficiency perspective. They conclude that the tension between stability and efficiency arises because the players do not account for the indirect negative effects that their connections bring to their neighbors.

Jackson and Wolinsky (1996) assume that a formation of a link between two agents require mutual consent of the agents. By contrast, Bala and Goyal (2000) weaken this assumption and allow agents to form links with others unilaterally by incurring the cost of the link. This modification in the modeling allows the authors to be able to use Nash equilibrium and its refinements in their analysis. They study both one-way and two-way flow of benefits. In the model with one-way flow, only the player who forms the link benefits from it, while in two-way flow, once the link is formed, both players enjoy the benefits. Moreover, similar to Jackson and Wolinsky (1996), there are also benefits from indirect communication from those to whom their adjacent links are connected. In

[2] Pairwise stability allows only requires only pairwise incentive conpatibility. Therefore, while analyzing for pairwise stability, deviations on a single link is considered at a time. However, there may be situations that coalitions larger than pairs can be formed. There are stronger notions of stability allowing for larger coalitions. For example, Dutta and Mutuswami (1997) study strong stability (coalition-proof Nash equilibrium) with a generalized version of Myerson (1977)'s model to see whether the tension between stability and efficiency can be resolved. Similar to Jackson and Wolinksy (1996)'s result that, due to symmetry, the tension between efficiency and stability still remains even if stronger notions of stability is applied, Dutta and Mutuswami (1997) conclude that the conflict between efficiency, stability and symmetry remains.

their benchmark model, the indirect communication is assumed to be frictionless, i.e., without decay. They also analyze the model with decay, in which the value of the benefit from the indirect links decays with the distance of the relationship, and in case of multiple paths, only the shortest path is considered. Moreover, like Jackson and Wolinsky (1996), Bala and Goyal (2000) analyze the model under the assumption that the agents are symmetric.

When there is no decay, Bala and Goyal (2000) show that Nash equilibrium is either the empty network or connected, that is, there exists a path between every pair of players. In the one-way flow model, strict Nash equilibrium structures are the empty network and the wheel network, in which a single directed cycle is formed with each player investing in exactly one link. Moreover, for a large set of parameters, the wheel is also the unique efficient architecture. Whereas, in the two-way flow model, strict Nash equilibrium structures are the empty network and the center-sponsored star network, in which one agent forms all the links. Furthermore, a star is also an efficient network for a class of payoff functions. Where there is decay, strict Nash networks are also connected. However, characterization of the strict Nash and efficient networks becomes difficult as the distances between the agents become relevant in computation of the indirect benefit. By focusing on low levels of decay, they obtain partial results. In one-way flow, the wheel and the star networks are strict Nash; while, in two-way flow model, the star is the unique efficient network and also a strict Nash equilibrium for a wide range of parameters.

There are three papers that drop the assumption of binary link strengths, and therefore, are closely related to our model. In the first one, Bloch and Dutta (2009) analyze a weighted link formation game in which players have fixed endowments to invest in relationships with others. In the baseline model, which is similar to our model with additively separable link strength function, they assume link strength is an additively separable and convex function of individual investments. However, unlike our model, agents use only the path which maximizes the product of link strengths, i.e., the most reliable path. They show that both the stable and efficient network architectures are stars. Moreover, they study the case were the agents' investments are perfect complements. Nonetheless, they could only provide a partial characterization of the stable and efficient structures as the analysis becomes intractable once the indirect benefits are taken into account.

On the other hand, Rogers (2006) considers a weighted link formation game in which all paths between agents are taken into account when calculating the indirect benefits. In this model, each player has an intrinsic positive base utility that would be his payoff in the absence of any network connections and an amount of time to allocate in forming relationships. Players are allowed to be heterogenous with respect to their intrinsic values and their time constraints. In addition to their intrinsic utility, players benefit from other players by interacting with each other. The more time a player spends on a player with higher intrinsic utility, the higher utility he obtains. Specifically, in Rogers' model, the benefit of forming a link is the product of the total value of the other agent and the strength of the relevant link. The total value of each agent is the sum of the benefits from all connections to other agents plus the agent's intrinsic utility. However, once a pair of agents are connected, they obtain each other's intrinsic utility with a weight of the link between them. Thus, this leads to multiple counting of an agent's intrinsic value in his total value. Apart from this, calculating the total value of agents in this way allows Rogers to separate the flow of benefits into "taking" and 'giving" components. That is, in giving model, the link decisions represent the giving of benefits, whereas, in taking model, the link decisions represent the taking of benefits. He finds that with exception of some Nash equilibria in giving model, all stable and efficient networks are identified as interior. Since heterogeneity between agents are allowed, both equilibrium and efficient networks display heterogeneity in link strengths. Moreover, by separating the flow of benefits, Rogers is able to provide new insights with respect to the efficiency of the stable network architectures. Particularly, the source of inefficiency is identified as the giving incentives and the inefficiency is present only when there exists heterogeneity among agents in terms of their budget constraints.

Lastly, another work analyzing heterogenous agents is by Brueckner (2006). Brueckner considers friendship networks concentrating on three player networks. He adopts a stochastic approach to link formation with the probability depending on the noncooperative investment. Even though this alternative approach leads to a simpler mathematical structure, the network architecture at equilibrium cannot be specified. Instead, his analysis focuses on the links which are most likely to form. Moreover, opposed to previous literature, he assumes that benefits are zero when more than two links are involved. Therefore, if agents i and j are connected, then agent i gains from socializing with j's direct friends but receives no benefits from j's indirect friendships. Brueckner shows that individual investment in friendship formation is too low. Moreover, in an asymmetric setting, friendship links involving attractive agent, who has personal magnetism or a broad group of acquaintances, are most likely to form.

3 The General Model of Link Formation

In this section, we present the general model of link formation game and prove the existence of the equilibrium before imposing further restrictions to the model. We also introduce the notation and definitions to be used throughout the rest of the paper.

Let $N = \{1, 2, 3..., n\}$ be the set of players. Each player j has information worth r_j and 1 unit of time to allocate across links to others. A strategy for player j's will be denoted z_j. It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set. A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $z_{-j} = (z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_n)$ denote the strategies of all the players except j.

Each player can benefit not only from his own information, but also from the information of other players if he is linked to them. The stronger the link is, the greater the share of information is transmitted. However, when information is transmitted along a link, it depreciates by some factor $0 < \beta \leq 1$. Thus, if the direct link between player i and player j has strength $\sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i. Moreover, we assume that links can transmit indirect information. Then, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i. However, following Brueckner (2006), our framework assumes that the benefits from indirect information transfers are zero when more than two links are involved.

We assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous, non-decreasing and concave in z_{ij} and z_{ji}. Moreover, we suppose $\sigma(0,0) = 0$ and $\sigma(1,1) = 1$ so that $\sigma_{ij} \in [0,1]$. In the following sections, we will work on with specific link strength functions.

Let $S_j(z)$ denote player j's payoff from strategy profile z. Since we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \right) r_i$$

For now, we will assume that player j can obtain less information exclusive to player i than player i. Specifically, we will assume

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} < 1 \tag{1}$$

as long as $0 < \beta < 1$. In the next sections, we will prove that (1) holds for each specified link strength function.

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$\begin{aligned} S_j(z) &= r_j + \sum_{i \neq j} S_{ij}(z) \\ &= r_j + \sum_{i \neq j} \left(\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \right) r_i \end{aligned}$$

We will use Nash equilibrium as the equilibrium concept throughout the paper. A network is Nash stable if there is no profitable deviations by individual agents.

Definition 1 *A strategy profile z is a Nash equilibrium if and only if, for all $j \in N$, it holds that*

$$S_j(z) \geq S_j(z_{-j}, z_j') \text{ for all } z_j' \in Z_j$$

The timeline for non-cooperative Nash equilibrium for n players is as follows: At time zero, players learn the value of each player's information, i.e., $r_j, j \in N$. Each player has 1 unit of time to invest in communication with other players. At time one, they simultaneously choose how much time to invest in other players. At time two, the players exchange information according to their strategies. Therefore, the optimization problem for each player j is as follows:

$$\text{maximize } S_j(z_j, z_{-j}) = r_j + \sum_{j \neq i} \left(\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} \right) r_i \text{ subject to}$$

$$z_j = \{z_{jk}\}_{k \neq j}$$

$$0 \leq z_{jk} \leq 1 \text{ for all } k \in N \setminus \{j\}$$

$$\sum_{k \neq j} z_{jk} = 1$$

Now, we will prove that the general game of link formation has an equilibrium. Notice that the players have infinite strategy sets; therefore, we will use Debreu, Fan, Glicksberg Theorem to prove the existence of the equilibrium.

Theorem 2 *(Debreu, Fan, Glicksberg) Consider a strategic form game* $\left\langle N, (Z_i)_{i \in N}, (S_i)_{i \in N} \right\rangle$ *with infinite strategy sets such that for each* $i \in N$ *:*

1. Z_i *is convex and compact.*

2. $S_i(z_i, z_{-i})$ *is continuous in* z_{-i}*.*

3. $S_i(z_i, z_{-i})$ *is continuous and quasiconcave in* z_i*.*

   The game has a pure strategy Nash equilibrium.

Proposition 3 *The general model of link formation game with n-players has a Nash equilibrium.*

Proof. The strategy set Z_i for player i is a simplex defined by

$$z_i = \{z_{ik}\}_{k \neq i}$$

$$0 \leq z_{ik} \leq 1 \text{ for all } k \in N \setminus \{i\}$$

$$\sum_{k \neq i} z_{ik} = 1$$

Thus, each Z_i for all $i \in N$ is convex and compact.

We assume that the general link strength function between players i and j, σ_{ij} is continuous and concave in z_{ij} and z_{ji}.Since the payoff function for player i, $S_i(z_i, z_{-i})$, is linear with respect to σ_{ij}, $S_i(z_i, z_{-i})$ is jointly continuous in both z_i and z_{-i}, and concave in z_i. Therefore, by Theorem 2, a pure strategy Nash equilibrium exists for the general model of link formation game with n-players.

■

Before we proceed to the analysis, we introduce formal definitions of the concepts to be used in the following chapters.

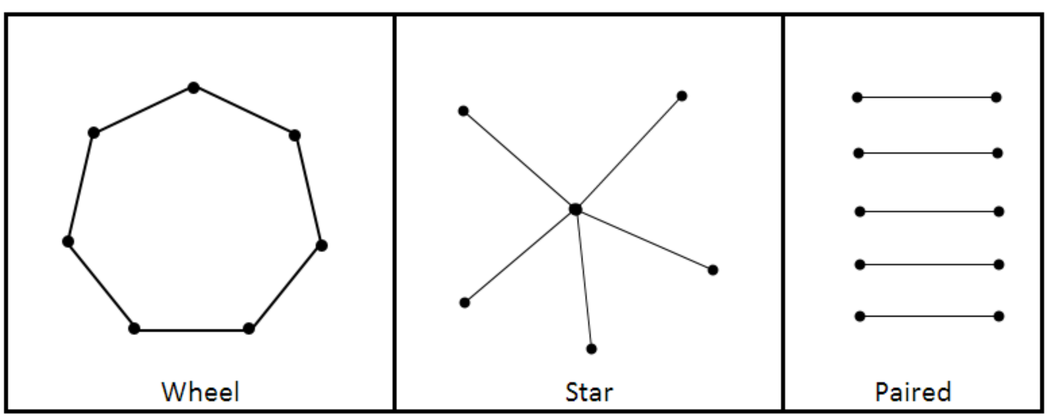
Definition 4 *A network (N, σ) consists of a set of nodes $N = \{1, 2, ..., n\}$ and a real-valued n x n matrix, σ, where σ_{ij} represents the link strengths between the players i and j. A path in a network (N, σ) between players i and j is a sequence of links $i_1 i_2; i_2 i_3; ...i_{K-1} i_K$ such that $\sigma_{i_{k-1} i_k} > 0$ for each $k \in \{1, 2, ...K\}$ with $i_1 = i$ and $i_K = j$, and such that each node in the sequence $i_1, i_2, ...i_K$ is distinct.*

Definition 5 *A network (N, σ) is connected if for each $i \in N$ and $j \in N$ there exists a path between i and j.*

Definition 6 *A component of a network (N, σ) is a nonempty subnetwork (N', σ') such that $\emptyset \neq N' \subset N$, $\sigma' \subset \sigma$ and*

- *(N', σ') is connected, and*
- *if $i \in N'$ and $ij \in \sigma$, then $j \in N'$ and $ij \in \sigma'$.*

The following are formal definitions of particular network architectures. These structures are illustrated in the following figure.



Definition 7 *A network (N, σ) is an n-player wheel if it consists of n directed links and has a single directed cycle that involves n players.*

Definition 8 *A network (N, σ) is a star if there exists a player i (the center of the star) such that all other players connected to the center. That is the player at the center has direct links to $n - 1$ players and each of the other players has only one direct link.*

Definition 9 *The player j is ostracized if $z_{ij} = 0$ for all $i \neq j$.*

Definition 10 *A network is said to be paired if for every player i (except one agent ostracized in the case that n is odd), there exists a player j such that*

$$z_{ij} = z_{ji} = 1$$

In the next sections, we will focus on special cases of our model in which we impose restrictions on link strength function. First, we will assume that the strength of the link is an additively separable function. This case allows players form links unilaterally to other players. Then, we will assume that the link strength function is Cobb-Douglas in which players have to have bilateral agreement to form links with each other.

4 Additively Separable Link Strength

In the previous section, we assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous and concave in z_{ij} and z_{ji}. In this section, we introduce further restriction and assume the link strength function between players i and j is

$$\sigma_{ij} = \frac{1}{2}z_{ij} + \frac{1}{2}z_{ji}$$

In other words, players i and j are perfect substitutes in terms of investment levels.

As σ_{ij} is linear in z_{ij} and z_{ji}, then player j's payoff is also continuous in all actions and concave in his own action s_i. Therefore, we know that there exists a pure strategy Nash equilibrium for the game as the conditions in Theorem 2 holds. Moreover, notice that this link strength function allows players to form links unilaterally to other players. The intuition behind this is that even if player i doesn't want to connect to player j, player j can still obtain some information of player i by spending time on researching on player i.

We will start our analysis with characterization of Nash equilibrium network structures with n-players where investments are perfect substitutes. These results will form the basis for comparison of the network structures under different link strength functions. Moreover, we present a complete characterization of the set of strong Nash equilibrium where the equilibrium network structure is a star.

Let $N = \{1, 2, 3..., n\}$ be the set of players. A strategy for player j's will be denoted z_j. It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set.

A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $S_j(z)$ denote player j's payoff from strategy profile z. If we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta \frac{1}{2} (z_{ij} + z_{ji}) + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} (z_{jk} + z_{kj}) \right) r_i$$

We have assumed in the previous section that player j can obtain less information exclusive to player i than player i. Specifically, the following inequality holds as long as $0 < \beta < 1$.

$$\beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} < 1$$

Now, we will show that if $\sigma_{ij} = \frac{1}{2} z_{ij} + \frac{1}{2} z_{ji}$ and $0 < \beta < 1$

$$\begin{aligned} \beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} &< 1 \\ \beta \frac{1}{2} (z_{ij} + z_{ji}) + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} (z_{jk} + z_{kj}) &< 1 \end{aligned}$$

As $0 \leq z_{jk} \leq 1$ for all $k \neq j$,

$$\sigma_{kj} = \frac{1}{2} (z_{jk} + z_{kj}) \leqslant 1$$

Therefore, we have

$$\begin{aligned}
\beta\frac{1}{2}\left(z_{ij}+z_{ji}\right)+\beta^{2}\sum_{k\neq i,j}\frac{1}{2}\left(z_{ik}+z_{ki}\right)\frac{1}{2}\left(z_{jk}+z_{kj}\right) &\leqslant \beta\frac{1}{2}\left(z_{ij}+z_{ji}\right)+\beta^{2}\sum_{k\neq i,j}\frac{1}{4}\left(z_{ik}+z_{ki}\right)\\
&= \beta\frac{1}{2}\left(z_{ij}+z_{ji}\right)+\beta^{2}\sum_{k\neq i,j}\frac{1}{4}z_{ik}+\beta^{2}\sum_{k\neq i,j}\frac{1}{4}z_{ki}\\
&= \beta\frac{1}{2}\left(z_{ij}+z_{ji}\right)+\beta^{2}\frac{1}{4}\left(1-z_{ij}\right)+\beta^{2}\frac{1}{4}\left(1-z_{ji}\right)\\
&= \left(\beta-\frac{\beta^{2}}{2}\right)\frac{1}{2}\left(z_{ij}+z_{ji}\right)+\frac{\beta^{2}}{2}
\end{aligned}$$

As $\sigma_{ij}\leqslant 1$ and $0<\beta<1$, we have

$$\beta\sigma_{ij}+\beta^{2}\sum_{k\neq i,j}\sigma_{ik}\sigma_{kj}\leqslant\beta<1$$

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$S_j(z)=r_j+\sum_{i\neq j}S_{ij}(z)$$

Let $z_{-j}=(z_1,z_2,...,z_{j-1},z_{j+1},...,z_n)$ denote the strategies of all the players except j. Let (z_{-j},z'_j) denote the strategy profile when all players except j choose according to z but player j chooses z'_j. A strategy profile z is a Nash equilibrium if and only if, for all $j\in N$, it holds that

$$S_j(z)\geq S_j(z_{-j},z'_j)\quad\text{for all } z'_j\in Z_j.$$

Let $\Omega(z)$ denote the set of players that some other player invests in under strategy profile z. That is

$$\Omega(z)=\left\{j:\ z_{ij}>0\text{ for some } i\right\}.$$

Since a player cannot invest in himself $\Omega(z)$ must contain at least two players.

Fix the strategies z_{-j} of all players except player j. Let z_j^a denote the strategy such that $z_{ja}=1$, i.e., player j invests only in player a. His payoff from this strategy is

$$
\begin{aligned}
S_j(z_{-j}, z_j^a) &= r_j + \sum_{i \neq j} \left(\beta \frac{1}{2} (z_{ij} + z_{ji}) + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} (z_{jk} + z_{kj}) \right) r_i \\
&= r_j + \left(\beta \frac{1}{2} (z_{aj} + 1) + \beta^2 \sum_{k \neq a,j} \frac{1}{2} (z_{ak} + z_{ka}) \frac{1}{2} z_{kj} \right) r_a \\
&\quad + \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} (1 + z_{aj}) + \beta^2 \sum_{k \neq i,j,a} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} z_{kj} \right) r_i \\
&= r_j + \beta \frac{1}{2} r_a + \left(\beta \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq a,j} \frac{1}{2} (z_{ak} + z_{ka}) \frac{1}{2} z_{kj} \right) r_a \\
&\quad + \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq i,j,a} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} z_{kj} \right) r_i \\
&\quad + \beta^2 \sum_{i \neq j,a} \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} r_i \\
&= r_j + \beta \frac{1}{2} r_a + \left(\beta \frac{1}{2} z_{aj} + \beta^2 \sum_{k \neq a,j} \frac{1}{2} (z_{ak} + z_{ka}) \frac{1}{2} z_{kj} \right) r_a \\
&\quad + \sum_{i \neq j,a} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} z_{kj} \right) r_i + \beta^2 \sum_{i \neq j,a} \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} r_i \\
&= r_j + \beta \frac{1}{2} r_a + \sum_{i \neq j} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} z_{kj} \right) r_i + \beta^2 \sum_{i \neq j,a} \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} r_i
\end{aligned}
$$

Similarly,

$$
S_j(z_{-j}, z_j^b) = r_j + \beta \frac{1}{2} r_b + \sum_{i \neq j} \left(\beta \frac{1}{2} z_{ij} + \beta^2 \sum_{k \neq i,j} \frac{1}{2} (z_{ik} + z_{ki}) \frac{1}{2} z_{kj} \right) r_i + \beta^2 \frac{1}{2} \sum_{i \neq j,b} (z_{ib} + z_{bi}) \frac{1}{2} r_i
$$

The payoff difference between only investing in player a and only investing in player b is therefore

$$
\begin{aligned}
& S_j(z_{-j}, z_j^a) - S_j(z_{-j}, z_j^b) && (2) \\
= \quad & \beta \frac{1}{2} (r_a - r_b) + \beta^2 \sum_{i \neq j,a} \frac{1}{2} (z_{ia} + z_{ai}) \frac{1}{2} r_i - \beta^2 \sum_{i \neq j,b} \frac{1}{2} (z_{ib} + z_{bi}) \frac{1}{2} r_i && (3)
\end{aligned}
$$

Lemma 11 *Consider any Nash equilibrium z and let σ denote the implied link-strengths. Suppose $\{a, b\} \subseteq \Omega(z)$, $h \neq b$ and $m \neq a$. Then if $z_{ha} > 0$ and $z_{mb} > 0$ the following inequality must hold:*

$$
(\sigma_{ha} - \sigma_{hb}) r_h + (\sigma_{mb} - \sigma_{ma}) r_m \leq 0 \tag{4}
$$

Proof. Suppose player h invests in $a \in \Omega(z)$, i.e., $z_{ha} > 0$. Then, by the Nash property, we must have

$$S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b) \geq 0.$$

Suppose some other player, say $m \neq h$, invests in $b \in \Omega(z)$ where $b \neq h$, i.e., $z_{mb} > 0$. By the Nash property, we must have

$$S_m(z_{-m}, z_m^b) - S_m(z_{-m}, z_m^a) \geq 0. \tag{5}$$

Applying (2) we find that

$$\begin{aligned}
& \left[S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b)\right] - \left[S_m(z_{-m}, z_m^a) - S_m(z_{-m}, z_m^b)\right] \\
= & \left[\beta \frac{1}{2}(r_a - r_b) + \beta^2 \sum_{i \neq h,a} \frac{1}{2}(z_{ia} + z_{ai}) \frac{1}{2} r_i - \beta^2 \sum_{i \neq h,b} \frac{1}{2}(z_{ib} + z_{bi}) \frac{1}{2} r_i\right] \\
& - \left[\beta \frac{1}{2}(r_a - r_b) + \beta^2 \sum_{i \neq m,a} \frac{1}{2}(z_{ia} + z_{ai}) \frac{1}{2} r_i - \beta^2 \sum_{i \neq m,b} \frac{1}{2}(z_{ib} + z_{bi}) \frac{1}{2} r_i\right] \\
= & \beta^2 \frac{1}{2}(z_{ma} + z_{am} - z_{mb} - z_{bm}) \frac{1}{2} r_m + \beta^2 \frac{1}{2}(z_{hb} + z_{bh} - z_{ha} - z_{ah}) \frac{1}{2} r_h
\end{aligned}$$

Therefore,

$$\begin{aligned}
& S_h(z_{-h}, z_h^a) - S_h(z_{-h}, z_h^b) \\
= & \left[S_m(z_{-m}, z_m^a) - S_m(z_{-m}, z_m^b)\right] + \frac{\beta^2}{4}(z_{ma} + z_{am} - z_{mb} - z_{bm}) r_m + \frac{\beta^2}{4}(z_{hb} + z_{bh} - z_{ha} - z_{ah}) r_h \\
\geq & 0
\end{aligned}$$

The term in square brackets is non-positive by (5). Therefore,

$$(z_{ma} + z_{am} - z_{mb} - z_{bm}) r_m + (z_{hb} + z_{bh} - z_{ha} - z_{ah}) r_h \geq 0$$

which implies that (4) holds. ■

Proposition 12 *Every player is connected to every other player either directly or indirectly with no more than two links under any Nash equilibrium z.*

Proof. First, we will prove that the network (N, σ) under the Nash equilibrium z is connected by contradiction. Then, we will show that each player is connected either directly or indirectly with no more than two links.

Suppose that the network (N, σ) under the Nash equilibrium z is not connected. Therefore, there exists a player i and a player j such that there is no path in (N, σ) between i and j. Moreover, since the network (N, σ) is not connected, there would be at least two components of (N, σ); (N', σ') and (N'', σ'') such that $\emptyset \neq N' \subset N$, $\emptyset \neq N'' \subset N$ and $N' \cap N'' = \emptyset$. Without loss of generality, assume that player i is in N' and player j is in N''. Since player i cannot invest in himself, there exists a player a such that $z_{ia} > 0$. Moreover, since (N', σ') is connected, $a \in N'$. Similarly, since player j cannot invest in himself, there exists a player b such that $z_{jb} > 0$ and since (N'', σ'') is connected, $b \in N''$. Since $a \in N'$ and $b \in N''$, we have $i \neq b$ and $j \neq a$. Then, from Lemma 11, as $\{a, b\} \subseteq \Omega(z)$, $i \neq b$ and $j \neq a$, and $z_{ia} > 0$ and $z_{jb} > 0$ the following inequality must hold:

$$(\sigma_{ia} - \sigma_{ib})\, r_i + (\sigma_{jb} - \sigma_{ja})\, r_j \leq 0$$

Since $\sigma_{ia} > 0$ and $\sigma_{jb} > 0$ as $z_{ia} > 0$ and $z_{jb} > 0$, we should have $\sigma_{ib} > 0$ and/or $\sigma_{ja} > 0$. Suppose $\sigma_{ib} > 0$, then there exists a path from player i to j via player b where $\sigma_{ib} > 0$ and $\sigma_{bj} > 0$. This leads to a contradiction with the assumption that the network (N, σ) under the Nash equilibrium z is not connected. Similar argument holds if we have $\sigma_{ja} > 0$ as there exists a path from player i to j via player a where $\sigma_{ia} > 0$ and $\sigma_{aj} > 0$. Therefore, the network (N, σ) under the Nash equilibrium z is connected.

Now, suppose that the shortest link between players i and j is greater than two. Since players i and j cannot invest in themselves, there exists players $a \neq i, j$ and $b \neq a, i, j$ such that $z_{ia} > 0$ and $z_{jb} > 0$. Notice that if $a = b$ or $a = j$ or $b = i$, the shortest link between players i and j becomes less than or equal to two. Then, from Lemma 11, as $\{a, b\} \subseteq \Omega(z)$, $i \neq b$ and $j \neq a$, and $z_{ia} > 0$ and $z_{jb} > 0$ the following inequality must hold:

$$(\sigma_{ia} - \sigma_{ib})\, r_i + (\sigma_{jb} - \sigma_{ja})\, r_j \leq 0$$

Since $\sigma_{ia} > 0$ and $\sigma_{jb} > 0$ as $z_{ia} > 0$ and $z_{jb} > 0$, we should have $\sigma_{ib} > 0$ and/or $\sigma_{ja} > 0$. Suppose $\sigma_{ib} > 0$, then there exists a path from player i to j via player b where $\sigma_{ib} > 0$ and $\sigma_{bj} > 0$. This leads to a contradiction with the assumption that the shortest link between players i and j is greater than two. Suppose $\sigma_{ja} > 0$, then there exists a path from player i to j via player a where $\sigma_{ia} > 0$ and $\sigma_{aj} > 0$. This leads to a contradiction with the assumption that the shortest link between players i and j is greater than two. Therefore, each player i and j should be connected either directly or indirectly with no more than two links. ■

Even though the most distinct feature of our model is the weighted link strength strategies, we will study strict Nash equilibria as a refinement in the next results. For this purpose, we will rewrite player j's payoff as

$$S_j(z) = \frac{\beta}{2} \sum_{i \neq j} z_{ji} \left(r_i + \beta \sum_{k \neq i,j} \sigma_{ik} r_k \right) + \phi(z_{-j})$$

where ϕ is a function that doesn't depend on player j's strategy z_j. As player j's objective function $S_j(z)$ is linear in the choice variables, z_{ji} for all $i \neq j$, for any given z_{-j}, player j maximizes his payoff assigning positive link strength only to the players that maximize

$$\pi_{ji} = r_i + \beta \sum_{k \neq i,j} \sigma_{ik} r_k$$

We will denote the unique maximizer of π_{ji} for player j as j^*. If there is a unique maximizer, j^*, then player j should set $z_{jj^*} = 1$.

Lemma 13 *Assume that in a Nash equilibrium z, for distinct players i and j, there exists a unique maximizer j^* of $\pi_{jk}, k \in N \setminus \{j\}$ for player j and there exists a unique maximizer i^* of $\pi_{ik}, k \in N \setminus \{i\}$ for player i. If $j \neq i^*$ and $i \neq j^*$, then $j^* = i^*$.*

Proof. Consider any Nash equilibrium z in which, for distinct players i and j, there exists a unique maximizer i^* of $\pi_{ik}, k \in N \setminus \{i\}$ for player i such that $i^* \neq j$, and player j chooses $z_{ja} = 1$ where $a \notin \{i, i^*\}$. Since $z_{ja} = 1$ and $z_{ia} = 0$, we have

$$\begin{aligned} \pi_{ja} &= r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} z_{ai} r_i \\ \pi_{ia} &= r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} (z_{aj} + 1) r_j \end{aligned}$$

Moreover, since $z_{ji^*} = 0$ and $z_{ii^*} = 1$, we have

$$\begin{aligned} \pi_{ji^*} &= r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} (z_{i^*i} + 1) r_i \\ \pi_{ii^*} &= r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} z_{i^*j} r_j \end{aligned}$$

Since player i^* is the unique maximizer of π_{ik} for $k \in N$, we have

$$\begin{aligned} \pi_{ii^*} &> \pi_{ia} \\ r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k + \frac{\beta}{2} z_{i^*j} r_j &> r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k + \frac{\beta}{2} (z_{aj} + 1) r_j \\ r_{i^*} + \beta \sum_{k \neq i^*,i,j} \sigma_{i^*k} r_k - \left(r_a + \beta \sum_{k \neq a,i,j} \sigma_{ak} r_k \right) &> \frac{\beta}{2} (z_{aj} + 1 - z_{i^*j}) r_j \end{aligned}$$

As $z_{aj}+1-z_{i^*j}\geqslant 0$, we should have

$$r_{i^*}+\beta\sum_{k\neq i^*,i,j}\sigma_{i^*k}r_k-\left(r_a+\beta\sum_{k\neq a,i,j}\sigma_{ak}r_k\right)>0$$

Then,

$$\pi_{ji^*}-\pi_{ja}=r_{i^*}+\beta\sum_{k\neq i^*,i,j}\sigma_{i^*k}r_k-\left(r_a+\beta\sum_{k\neq a,i,j}\sigma_{ak}r_k\right)+\frac{\beta}{2}\left(z_{i^*i}+1-z_{ai}\right)r_i$$

As $z_{i^*i}+1-z_{ai}\geqslant 0$, we have

$$\pi_{ji^*}-\pi_{ja}>0$$

This means player j is not best responding by choosing $z_{ja}=1$. This means the only candidates for j^* are i and i^*. By the hypothesis in the lemma, $j^*\neq i$. Hence, $j^*=i^*$ ■

Lemma 14 *In a strict Nash equilibrium z, if $j^*=i$ and $i^*=h\neq j$, then $h^*=i$.*

Proof. Choose a strategy profile in which each player $l\neq j$ sets $z_{ll^*}=1$ with $j^*=i$ and $i^*=h\neq j$. From Lemma 13, as $h\neq i$, we must have $h^*\in\{j,i\}$. Assume $z_{hj}=1$ and consider the following expressions:

$$\begin{aligned}
\pi_{ji} &= r_i+\beta\sum_{k\neq h,i,j}\sigma_{ik}r_k+\frac{\beta}{2}r_h\\
\pi_{jh} &= r_h+\beta\sum_{k\neq h,i,j}\sigma_{hk}r_k+\frac{\beta}{2}r_i\\
\pi_{ih} &= r_h+\beta\sum_{k\neq h,i,j}\sigma_{hk}r_k+\frac{\beta}{2}r_j\\
\pi_{ij} &= r_j+\beta\sum_{k\neq h,i,j}\sigma_{jk}r_k+\frac{\beta}{2}r_h
\end{aligned}$$

Since $\pi_{ji}-\pi_{jh}>0$ and $\pi_{ih}-\pi_{ij}>0$, we have

$$\begin{aligned}
\pi_{ji}-\pi_{ij} &> \pi_{jh}-\pi_{ih}\\
r_i+\beta\sum_{k\neq h,i,j}\sigma_{ik}r_k+\frac{\beta}{2}r_h-\left(r_j+\beta\sum_{k\neq h,i,j}\sigma_{jk}r_k+\frac{\beta}{2}r_h\right) &> \frac{\beta}{2}r_i-\frac{\beta}{2}r_j\\
r_i+\beta\sum_{k\neq h,i,j}\sigma_{ik}r_k+\frac{\beta}{2}r_j &> r_j+\beta\sum_{k\neq h,i,j}\sigma_{jk}r_k+\frac{\beta}{2}r_i\\
\pi_{hi} &> \pi_{hj}
\end{aligned}$$

Therefore, player h is not best responding if he sets $z_{hj}=1$. Hence, we have $h^*=i$. ■

Proposition 15 *Assume $r_{k-1} > r_k$ for all $k \in N \backslash \{1\}$. There are two sets of strict Nash equilibria of the simultaneous move game, both resulting in a star network. The first set of equilibrium strategies is given by*

$$\begin{aligned} z_{i1} &= 1 \text{ for all } i \in N \setminus \{1\} \\ z_{12} &= 1 \end{aligned}$$

Moreover, we have

$$S_1 > S_2 > ... > S_N$$

The second set of equilibrium strategies is given by

$$\begin{aligned} z_{ii^*} &= 1 \text{ for all } i \in N \setminus \{i^*\} \text{ where } i^* \neq 1 \\ z_{i^*1} &= 1 \end{aligned}$$

Moreover, we have

$$S_{i^*} - r_{i^*} > S_1 - r_1$$

$$S_1 > S_2 > S_{i^*-1} > S_{i^*+1} ... > S_N$$

For the second case to be strict Nash equilibrium, we need

$$\begin{aligned} r_{i^*} &> r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} \text{ where } j = \arg \max_{k \neq 1, i^*} r_k \\ r_{i^*} &> r_2 - \frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1, i^*, 2} \frac{r_k}{2} \text{ if } r_2 > r_{i^*} \end{aligned}$$

Proof. In a strict Nash equilibrium z, every j has a unique maximizer of $\pi_{ji}, i \in N \setminus \{j\}$ for fixed z_{-j}. Then, player j sets $z_{jj^*} = 1$ in any strict Nash equilibrium. By Lemma 13, we know that for distinct players i and j, if $j \neq i^*$ and $i \neq j^*$, then $j^* = i^*$. This restricts the relationships between two distinct players into two: the players are either directly connected or they are connected to the same player. Moreover, if the players are directly connected, then Lemma 14 eliminates any wheel structure in the equilibrium. Therefore, by Lemma 13 and 14, the only possible strategy profile for a strict Nash equilibrium is the following: There exists a player i^* such that

$$\begin{aligned} z_{ji^*} &= 1 \text{ for all } j \in N \setminus \{i^*\} \\ z_{i^*h} &= 1 \text{ where } h = \arg\max_{k \neq i^*} r_k. \end{aligned}$$

Now, consider the first set of equilibrium strategies given by

$$\begin{aligned} z_{i1} &= 1 \text{ for all } i \in N \setminus \{1\} \\ z_{12} &= 1 \end{aligned}$$

Player 1 is at the center of the network, i.e., $z_{i1} = 1$ for all $i \in N \setminus \{1\}$. Since,

$$\begin{aligned} \pi_{1i} &= r_i \text{ for all } i \neq 1 \\ 2 &= \arg\max_{k \neq 1} r_k \end{aligned}$$

player 1 is strictly best responding by choosing $z_{12} = 1$.

Players $j \in N \setminus \{1, 2\}$ are best responding since

$$r_1 + \beta r_2 + \beta \sum_{k \neq 1,2,j} \frac{r_k}{2} > r_2 + \beta r_1 > r_k + \frac{\beta}{2} r_1 \text{ for all } k \neq 1, 2, j$$

Finally, player 2 is best responding since

$$r_1 + \beta \sum_{k \neq 1,2} \frac{r_k}{2} > r_k + \frac{\beta}{2} r_1$$

for all $k \neq 1, 2$. Under this equilibrium strategies, the surplus for each player is

$$\begin{aligned} S_1 &= r_1 + \beta r_2 + \beta \sum_{k \neq 1,2} \frac{r_k}{2} \\ S_2 &= r_2 + \beta r_1 + \beta^2 \sum_{k \neq 1,2} \frac{r_k}{2} \\ S_n &= r_n + \frac{\beta}{2} r_1 + \beta^2 \sum_{k \neq 1,n} \frac{r_k}{2} \text{ for all } n \in N \setminus \{1, 2\} \end{aligned}$$

Therefore, we have

$$S_1 > S_2 > ... > S_N$$

Now, consider the second set of equilibrium strategies given by

$$\begin{aligned} z_{ii^*} &= 1 \text{ for all } i \in N \setminus \{i^*\} \\ z_{i^*1} &= 1 \end{aligned}$$

Player i^* is at the center of the network, i.e., $z_{ii^*} = 1$ for all $i \in N \setminus \{i^*\}$. Since,

$$\begin{aligned} \pi_{i^*i} &= r_i \text{ for all } i \neq i^* \\ 1 &= \arg\max_{k \neq i^*} r_k \end{aligned}$$

player i^* is strictly best responding by choosing $z_{i^*1} = 1$.

Players $j \in N \setminus \{i^*, 1\}$ are best responding if

$$\begin{aligned} r_{i^*} + \beta r_1 + \beta \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> r_1 + \beta r_{i^*} \\ \beta \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> (r_1 - r_{i^*})(1 - \beta) \\ \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} &> r_1 - r_{i^*} \\ r_{i^*} &> r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, j} \frac{r_k}{2} \end{aligned}$$

Since this holds for all players $j \in N \setminus \{i^*, 1\}$, we should have

$$r_{i^*} > r_1 - \frac{\beta}{1-\beta} \sum_{k \neq 1, i^*, h} \frac{r_k}{2} \text{ where } h = \arg\max_{k \neq 1, i^*} r_k$$

Finally, when $r_{i^*} < r_2$, player 1 is best responding if

$$\begin{aligned} r_{i^*} + \frac{\beta}{2} r_2 + \beta \sum_{k \neq 1, 2, i^*} \frac{r_k}{2} &> r_2 + \frac{\beta}{2} r_{i^*} \\ \beta \sum_{k \neq 1, 2, i^*} \frac{r_k}{2} &> (r_2 - r_{i^*}) \left(1 - \frac{\beta}{2}\right) \\ \frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1, 2, i^*} \frac{r_k}{2} &> r_2 - r_{i^*} \\ r_{i^*} &> r_2 - \frac{\beta}{1 - \frac{\beta}{2}} \sum_{k \neq 1, 2, i^*} \frac{r_k}{2} \end{aligned}$$

Under this equilibrium strategies, the surplus for each player is given by

$$\begin{aligned} S_1 &= r_1 + \beta r_{i^*} + \beta^2 \sum_{k \neq 1, i^*} \frac{r_k}{2} \\ S_{i^*} &= r_{i^*} + \beta r_1 + \beta \sum_{k \neq 1, i^*} \frac{r_k}{2} \\ S_n &= r_n + \frac{\beta}{2} r_{i^*} + \beta^2 \sum_{k \neq i^*, n} \frac{r_k}{2} \text{ for all } N \setminus \{1, i^*\} \end{aligned}$$

Therefore, we have

$$S_{i^*} - r_{i^*} > S_1 - r_1$$

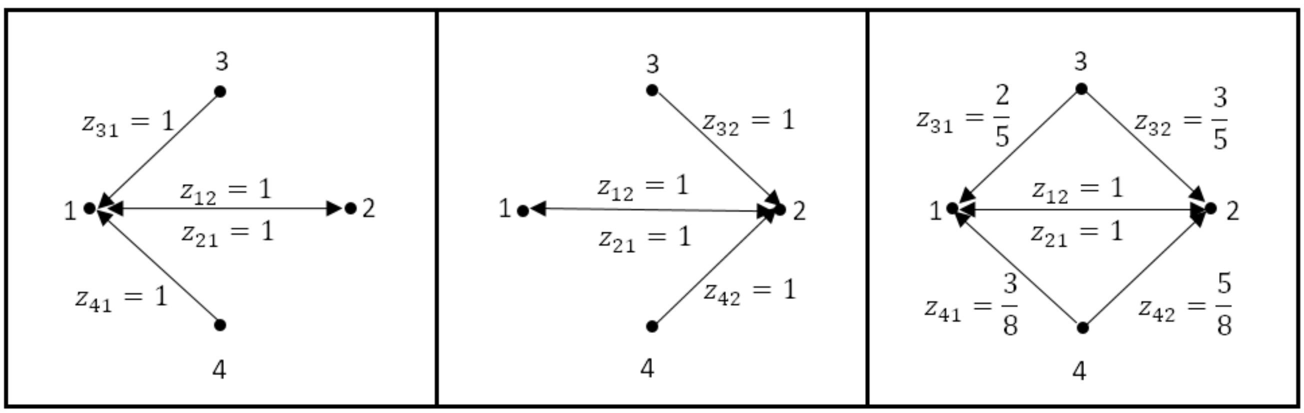
and

$$S_1 > S_2 > S_{i^*-1} > S_{i^*+1} ... > S_N$$

■

Even though we know that star networks are the only strict Nash equilibria of the simultaneous move game under additively separable link strength function, there may be Nash equilibrium with different network architecture when $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. Consider the following example with four players.

Example 16 *Assume that* $r_1 = 100, r_2 = 99, r_3 = 5, r_4 = 4$ *and* $\beta = \frac{2}{3}$*. As the intrinsic values of players 1 and 2 are very high compared to the other two, under any Nash equilibrium, 1 and 2 would invest all their time to each other. Then, there will be three different Nash equilibrium, in one of which players 3 and 4 allocate their time both on player 1 and 2. The following figure shows the investment levels at the Nash equilibrium. As opposed to the first two architecture, the network structure in the third Nash equilibrium is not a star.*



Proposition 17 *Nash equilibrium under additively separable link strength function may not be surplus maximizing outcome.*

Proof. We prove the result by using three player game. Full characterization of Nash equilibrium and surplus maximizing outcome is provided in the Appendix.

Assume that there are only three players and they are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. Under these conditions, there is only one Nash equilibrium[3] where $z_{12} = 1, z_{21} = 1, z_{31} = 1$. We will show that a higher level of total surplus could be achieved under

[3] Please refer to Proposition 24 in the Appendix for formal proof.

some conditions by changing the equilibrium strategy for either player 1 or player 3. The following shows the partial derivative of total surplus with respect to z_{12} evaluated at $z_{12} = 1, z_{21} = 1, z_{31} = 1$.

$$\left[\frac{\partial S}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta(r_3(2+\beta) - r_2(2-\beta))$$

If the difference between the intrinsic values of player 2 and 3 is low enough, that is, when

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

the partial derivative of the total surplus with respect to z_{12} would be negative at Nash Equilibrium. Then, a higher total social surplus could be achieved by decreasing z_{12} by a small amount so that player 1 allocate his time between player 2 and 3. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 3's surplus being higher than the total decrease in player 1 and 2's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{2}\beta\left(r_2 - r_3\right)$$

$$\left[\frac{\partial S_2}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(2r_1 - \beta r_3\right)$$

$$\left[\frac{\partial S_3}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta\left(2r_1 + \beta r_2\right)$$

On the other hand, the following shows the partial derivative of total surplus with respect to z_{31} evaluated at $z_{12} = 1, z_{21} = 1, z_{31} = 1$.

$$\left[\frac{\partial S}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(r_1(2-3\beta) - r_2(2-\beta)\right)$$

If the difference between the intrinsic values of player 1 and 2 is low enough, that is, when

$$\frac{r_1}{r_2} < \frac{2-\beta}{2-3\beta}$$

the partial derivative of the total surplus with respect to z_{31} would be negative at Nash Equilibrium. Then, a higher total social surplus could be achieved by decreasing z_{31} by a small amount so that player 3 allocate his time between player 1 and 2. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 2's surplus being higher than the total decrease in player 1 and 3's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{4}\beta\left(2(1-\beta)r_3 - \beta r_2\right)$$

$$\left[\frac{\partial S_2}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\frac{1}{4}\beta\left(2(1-\beta)r_3 + \beta r_1\right)$$

$$\left[\frac{\partial S_3}{\partial z_{31}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \frac{1}{2}\beta\left(r_1 - r_2\right)(1-\beta)$$

■

5 Cobb-Douglas Link Strength

In the general model, we assume that the general link strength function between players i and j

$$\sigma(z_{ij}, z_{ji}) = \sigma_{ij}$$

is continuous and concave in z_{ij} and z_{ji}. In this section, we assume the link strength function between players i and j is

$$\sigma_{ij} = z_{ij}z_{ji}$$

As σ_{ij} is continuous and linear in z_{ij} and z_{ji}, then player j's payoff is also continuous in all actions and concave in his own action s_i. Therefore, we know that there exists a pure strategy Nash equilibrium for the game as the conditions in Theorem 2 holds. Moreover, notice that this link strength function allows players to form links only when there is bilateral agreement between the players as opposed to the additively separable link strength function, in which player could form links unilaterally. The intuition behind this is the players have exclusive information and it can be shared only if they invest time to each other.

Unfortunately, the analysis of Nash equilibria with n-players under Cobb-Douglas link strength function becomes intractable with the inclusion of the indirect benefits into the model. Thus, we will focus only on a refinement to Nash equilibrium. We consider a sequential game with perfect information in which the players announce their strategies according to a random ordering. We show that there is a unique subgame perfect equilibrium of the sequential game which is also Nash equilibrium of the simultaneous move game.

Let $N = \{1, 2, 3..., n\}$ be the set of players. A strategy for player j's will be denoted z_j. It consists of his investment levels in the other players:

$$z_j = \{z_{jk}\}_{k \neq j}$$

and must satisfy

$$0 \leq z_{jk} \leq 1$$

for all $k \neq j$ and

$$\sum_{k \neq j} z_{jk} = 1.$$

Let Z_j denote player j's strategy set. A strategy profile consists of a strategy for each player. A strategy profile will be written

$$z = (z_1, z_2, ..., z_n) \in Z \equiv Z_1 \times Z_2 \times ... \times Z_n$$

Let $S_j(z)$ denote player j's payoff from strategy profile z. The strength of the link between i and j is

$$\sigma(z_{ij}, z_{ji}) = z_{ij} z_{ji}$$

Player j's information is worth r_j. Thus, if the direct link between player i and player j has strength $\sigma_{ij} = \sigma(z_{ij}, z_{ji})$, then obtaining player j's information via this direct link is worth $\beta\sigma(z_{ij}, z_{ji})r_j$ to player i where $0 < \beta < 1$. Moreover, if player j is linked to player k, then player k's information can be indirectly transmitted from k to i via j. Obtaining player k's information via this indirect link is worth $\beta\sigma(z_{ij}, z_{ji})\beta\sigma(z_{jk}, z_{kj})r_k$ to player i.

If we assume information can be transmitted by a chain of no more than two links, then the total amount of information that player j receives from player i, directly and indirectly, is

$$S_{ij}(z) = \left(\beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} \right) r_i$$

We have assumed that player j can obtain less information exclusive to player i than player i while introducing the general model. Specifically, the following inequality holds as long as $0 < \beta < 1$.

$$\beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} < 1$$

Now, we will show that if $\sigma_{ij} = \beta z_{ij} z_{ji}$ and $0 < \beta < 1$

$$\begin{aligned} \beta\sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik}\sigma_{kj} &< 1 \\ \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} &< 1 \end{aligned}$$

As $0 \leq z_{jk} \leq 1$ for all $k \neq j$,we have

$$\begin{aligned}
\beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} &\leqslant \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{jk} \\
&\leqslant \beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} (1 - z_{ij}) z_{jk} \\
&= \beta z_{ij} z_{ji} + \beta^2 (1 - z_{ij}) \sum_{k \neq i,j} z_{jk} \\
&= \beta z_{ij} z_{ji} + \beta^2 (1 - z_{ij}) (1 - z_{ji}) \\
&< \beta z_{ij} z_{ji} + \beta (1 - z_{ij}) (1 - z_{ji}) \\
&\leqslant \beta z_{ji} + \beta (1 - z_{ji})
\end{aligned}$$

Therefore, if $0 < \beta < 1$, we have

$$\beta \sigma_{ij} + \beta^2 \sum_{k \neq i,j} \sigma_{ik} \sigma_{kj} \leqslant \beta < 1$$

Player j's payoff $S_j(z)$ will be his own information plus the total amount of information he receives from others:

$$\begin{aligned}
S_j(z) &= r_j + \sum_{i \neq j} S_{ij}(z) \\
&= r_j + \sum_{i \neq j} \left(\beta z_{ij} z_{ji} + \beta^2 \sum_{k \neq i,j} z_{ik} z_{ki} z_{jk} z_{kj} \right) r_i
\end{aligned}$$

Let $z_{-j} = (z_1, z_2, ..., z_{j-1}, z_{j+1}, ..., z_n)$ denote the strategies of all the players except j. Let (z_{-j}, z'_j) denote the strategy profile when all players except j choose according to z but player j chooses z'_j. Let $\Omega(z)$ denote the set of players that some other player invests in under strategy profile z. That is

$$\Omega(z) = \{ j : \; z_{ij} > 0 \text{ for some } i \}.$$

Since a player cannot invest in himself $\Omega(z)$ must contain at least two players.

Fix the strategies z_{-j} of all players except player j. Let z_j^a denote the strategy such that $z_{ja} = 1$, i.e., player j invests only in player a. His payoff from this strategy is

$$S_j(z_{-j}, z_j^a) = r_j + \beta z_{ja} r_a + \beta^2 z_{ja} \sum_{k \neq, j, a} z_{ak} z_{ka} r_k$$

Proposition 18 *Assume $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$. There exists a Nash equilibrium of the simultaneous move game under the following strategies. For every player i, there exists a player j such that*

$$z_{ij} = z_{ji} = 1 \textit{ if } n \textit{ is even}$$

and if n is odd, there will be a player $l \neq 1$ that is ostracized. If n is odd then let

$$z_{lm} = 1 \textit{ where } z_{m1} = z_{1m} = 1$$

Proof. Suppose, we have even number of players. Assume that for every player i, there exists a player j such that

$$z_{ij} = z_{ji} = 1 \text{ if } n \text{ is even}$$

holds except player h, where h is paired with p. Thus, the surplus for player h is

$$S_h(z_{-h}, z_h) = r_h + \beta z_{jp} r_p$$

Notice that this surplus is maximized when $z_{jp} = 1$ and $z_{ji} = 0$ where $i \in N \setminus \{j, p\}$ as

$$S_j = r_j + \beta r_p \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{jp} r_p \text{ for all } z_j' \in Z_j$$

and $0 \leqslant z_{jp} \leqslant 1$.This holds for any $j \in N$.
Now, suppose we have odd number of players. Then, there will be a player $l \neq 1$ that is ostracized. Assume that

$$z_{lm} = 1 \text{ where } z_{m1} = z_{1m} = 1$$

Since each player is matched with another player according to the strategy profile above, none of the players spend time with player l. The players whose pair has information less than r_l would want to be paired with the player l. However, since player l's strategy is $z_{lm} = 1$, none of these players have an incentive to deviate. Thus, there is no incentive to deviate for players $i \in N \setminus \{l, m\}$. Moreover, player m has no incentive to deviate as he is paired with player 1. Let's check if player l has an incentive to deviate. As none of the players spend time with player l, the payoff for player l is

$$S_l = r_l$$

Since S_l doesn't depend on z_l, there is no incentive to deviate for player l.Therefore, we have an equilibrium. ■

Corollary 19 *Assume* $r_{k-1} > r_k$ *for all* $k \in N \setminus \{1\}$. *There exists a Nash equilibrium of the simultaneous move game where*

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \textit{ for all } i \in N \setminus \{n\} \textit{ where } i : odd. \\ z_n &: free \textit{ if } n : odd \end{aligned}$$

Therefore, at the equilibrium, we have

$$S_1 > S_2 > ... > S_n$$

This equilibrium will be referred to as "Assortative Pair Equilibrium (APE)".

Proof. Suppose, we have even number of players. Assume that

$$z_{i(i+1)} = z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd.$$

holds except player j. Assume that $j : odd$. Thus, the surplus for player j is

$$S_j(z_{-j}, z_j) = r_j + \beta z_{j(j+1)} r_{j+1}$$

Notice that this surplus is maximized when $z_{j(j+1)} = 1$ and $z_{ji} = 0$ where $i \in N \setminus \{j, j+1\}$ as

$$S_j = r_j + \beta r_{j+1} \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{j(j+1)} r_{j+1} \text{ for all } z_j^{\prime} \in Z_j$$

and $0 \leqslant z_{j(j+1)} \leqslant 1$.
Assume that $j : even$. Thus, the surplus for player j is

$$S_j(z_{-j}, z_j) = r_j + \beta z_{(j-1)j} r_{j-1}$$

Notice that this surplus is maximized when $z_{j(j+1)} = 1$ and $z_{ji} = 0, i \in N \setminus \{j, j+1\}$ as

$$S_j = r_j + \beta r_{j-1} \geqslant S_j(z_{-j}, z_j^{'}) = r_j + \beta z_{(j-1)j} r_{j-1} \text{ for all } z_j^{\prime} \in Z_j$$

and $0 \leqslant z_{(j-1)j} \leqslant 1$.This holds for any $j \in N$.
Now, suppose we have odd number of players. Assume that

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd. \\ z_n &: free \end{aligned}$$

We know from above that there is no incentive to deviate for players $i \in N \setminus \{n\}$.Let's check if player n has an incentive to deviate. Since each player with odd rank in terms of payoff ordering is

matched with the player that are subsequent to themselves according to the strategy profile above, none of the players spend time with player n. This makes the payoff for player n, $S_n = r_n$. Since S_n doesn't depend on z_n, there is no incentive to deviate for player n.
Thus, at equilibrium, we have

$$\begin{aligned} S_i &= r_i + \beta r_{i+1} \text{ if } i : odd \\ S_i &= r_i + \beta r_{i-1} \text{ if } i : even \\ S_n &= r_n \text{ if } n : odd \end{aligned}$$

If $i : odd$, then

$$S_i = r_i + \beta r_{i+1} > S_{i+1} = r_{i+1} + \beta r_i$$

as $0 < \beta < 1$. If $i : even$, then

$$S_i = r_i + \beta r_{i-1} > S_{i+1} = r_{i+1} + \beta r_{i+1}$$

as $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$ .Thus, we have

$$S_1 > S_2 > ... > S_n$$

■

Notice that when there are even number of players, under APE, each player with odd ranking will be matched with the player that comes after him according to the information ranking. That is, players will be matched in pairs according to their information levels, and they will form links to people who have similar level of information as themselves. On the other hand, if there are odd number of players, the player n who has the least valuable information will not be linked to anyone. Thus, player n will be ostracized and his choice variable z_n will be free.

Consider a sequential game with perfect information. An ordering of the players is chosen randomly at time zero. Then, at time 1, the first player according to the ordering chosen at time zero chooses his links publicly. Then, at time 2, after observing the first player's choices, the second player according to the random ordering chooses his links publicly. Similarly, at time n, after observing the choices of the previous players, player n according to the random ordering chooses his links. After each player choose their links, at time $n + 1$, the players exchange information according to their strategies.

Definition 20 *A Nash equilibrium outcome (of the simultaneous move game) is "robust" if there exists SOME ordering of the players, such that the subgame perfect equilibrium of the sequential move game with that ordering, generates that same outcome.*

Definition 21 *A Nash equilibrium outcome (of the simultaneous move game) is "strongly robust" if for ALL possible orderings of the players, the subgame perfect equilibrium of the sequential move game with that ordering, generates that same outcome.*

Proposition 22 *Assuming $r_{k-1} > r_k$ for all $k \in N \setminus \{1\}$, Assortative Pair Equilibrium of the simultaneous move game where*

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \textit{ for all } i \in N \setminus \{n\} \ \textit{ where } i : odd. \\ z_n &: free \textit{ if } n : odd \end{aligned}$$

is strongly robust.

Proof. Proof will be done in steps.

(Step One) Assume that player j is not ostracized. If $z_{ij} = 0$, then $z_{ji} = 0$ at equilibrium for the simultaneous and sequential move games.

Proof. Assume $z_{ij} = 0$. Since player j is not ostracized, there exists a player h such that $z_{hj} \neq 0$. Moreover, since $\sum_{k \neq j} z_{jk} = 1$, we can write $z_{jh} = 1 - z_{ji} - \sum_{k \neq j,i,h} z_{jk}$. We can substitute this into S_j. If we take the derivative of S_j with respect to z_{ji}, we get

$$\frac{\partial S_j}{\partial z_{ji}} = -\beta z_{hj} r_h - \beta^2 z_{hj} \sum_{k \neq j,h} z_{hk} z_{kh} r_k < 0$$

Thus, $z_{ji} = 0$.

■

(Step Two) If player 2 moves before player 1 and sets $z_{21} = 1$, then player 1 will set $z_{12} = 1$ at equilibrium for the sequential move game.

Proof. Suppose player 2 moves before player 1 in sequential game and sets $z_{21} = 1$. We can substitute $z_{13} = 1 - z_{12} - \sum_{k \neq 1,2,3} z_{1k}$ in player 1's payoff function and take derivative with respect to z_{12}

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k$$

Since $0 \leqslant z_{k3} \leqslant 1$ and $r_k < r_3$ for $k > 3$, we have $z_{k3} r_k < r_3$. Thus,

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k > \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} r_3 \sum_{k \neq 1,2,3} z_{3k}$$

Notice that $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31} - z_{32}$. From Step 1, we also know that $z_{32} = 0$ as $z_{23} = 0$ if player 3 is not ostracized. So, $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31}$ and we get

$$\begin{aligned}
\frac{\partial S_1}{\partial z_{12}} &> \beta r_2 - \beta z_{31} r_3 - \beta^2 (1 - z_{31}) z_{31} r_3 \\
&> \beta r_2 - \beta z_{31} r_3 - \beta^2 (1 - z_{31}) r_3 \\
&> \beta r_2 - \beta z_{31} r_3 - \beta (1 - z_{31}) r_3 \\
&> \beta r_2 - \beta r_3 > 0
\end{aligned}$$

Therefore, player 1 chooses $z_{12} = 1$.

If player 3 is ostracized, then $z_{k3} = 0$ for all $k \neq 1, 2, 3$. Then,

$$\frac{\partial S_1}{\partial z_{12}} = \beta r_2 - \beta z_{31} r_3 - \beta^2 z_{31} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k = \beta r_2 - \beta z_{31} r_3 > 0$$

Therefore, player 1 chooses $z_{12} = 1$.

■

(Step Three) If player 2 moves before player 1, then he sets $z_{21} = 1$.

Proof. Fix the strategies z_{-2} of all players except player 2. Let z_2^a denote the strategy such that $z_{2a} = 1$, i.e., player 2 invests only in player a, where $a \neq 1$. His payoff from this strategy is

$$S_2(z_{-2}, z_2^a) = r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} \sum_{k \neq 2,a} z_{ak} z_{ka} r_k$$

Moreover, he knows from the previous step that if he sets $z_{21} = 1$, then player 1 sets $z_{12} = 1$. His payoff from this strategy is

$$S_2(z_{-2}, z_2^1) = r_2 + \beta r_1$$

Notice that since $r_a < r_1$, we have

$$\begin{aligned}
S_2(z_{-2}, z_2^a) &= r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} \sum_{k \neq 2,a} z_{ak} z_{ka} r_k < r_2 + \beta z_{a2} r_a + \beta^2 z_{a2} r_1 \sum_{k \neq 2,a} z_{ak} \\
&= r_2 + \beta z_{a2} r_a + \beta^2 (1 - z_{a2}) z_{a2} r_1 < r_2 + \beta r_1 = S_2(z_{-2}, z_2^1)
\end{aligned}$$

Therefore, player 2 cannot get higher payoff by linking to someone rather than player 1. This also implies that player 2 wouldn't use mixed strategies such as $0 < z_{2i} < 1$ for some $i \in N \setminus \{1\}$. To see this, suppose player 2 invests in $a \in \Omega(z)$, i.e., $z_{2a} > 0$. Then, by the Nash property, we must have

$$S_2(z_{-2}, z_2^a) - S_2(z_{-2}, z_2^b) \geq 0$$

for all $b \in N$. However, we know that

$$S_2(z_{-2}, z_2^a) < S_2(z_{-2}, z_2^1)$$

for $a \in N \setminus \{1\}$. Thus, if player 2 moves before player 1, he sets $z_{21} = 1$.

■

(Step Four) If player 1 moves before player 2 and sets $z_{12} = 1$, then player 2 will set $z_{21} = 1$ at equilibrium for the sequential move game.

Proof. Suppose player 1 moves before player 2 in sequential game and sets $z_{12} = 1$. We can substitute $z_{23} = 1 - z_{21} - \sum_{k \neq 1,2,3} z_{2k}$ in player 2's payoff function and take derivative with respect to z_{21}

$$\frac{\partial S_2}{\partial z_{21}} = \beta r_1 - \beta z_{32} r_3 - \beta^2 z_{32} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k$$

Since $0 \leqslant z_{k3} \leqslant 1$ and $r_k < r_3$ for $k > 3$, we have $z_{k3} r_k < r_3$. Thus,

$$\frac{\partial S_2}{\partial z_{21}} = \beta r_1 - \beta z_{32} r_3 - \beta^2 z_{32} \sum_{k \neq 1,2,3} z_{3k} z_{k3} r_k > \beta r_2 - \beta z_{32} r_3 - \beta^2 z_{32} r_3 \sum_{k \neq 1,2,3} z_{3k}$$

Notice that $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31} - z_{32}$. From Step One, we also know that $z_{31} = 0$ as $z_{13} = 0$. So, $\sum_{k \neq 1,2,3} z_{3k} = 1 - z_{31}$ and we get

$$\begin{aligned}
\frac{\partial S_2}{\partial z_{21}} &> \beta r_1 - \beta z_{32} r_3 - \beta^2 (1 - z_{32}) z_{32} r_3 \\
&> \beta r_1 - \beta z_{32} r_3 - \beta^2 (1 - z_{32}) r_3 \\
&> \beta r_1 - \beta z_{32} r_3 - \beta (1 - z_{32}) r_3 \\
&> \beta r_1 - \beta r_3 > 0
\end{aligned}$$

Therefore, player 2 chooses $z_{21} = 1$.

■

(Step Five) If player 1 moves before player 2, then he sets $z_{12} = 1$.

Proof. Fix the strategies z_{-1} of all players except player 1. Let z_1^a denote the strategy such that $z_{1a} = 1$, i.e., player 1 invests only in player a, where $a \neq 2$. His payoff from this strategy is

$$S_1(z_{-1}, z_1^a) = r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} \sum_{k \neq 1,a} z_{ak} z_{ka} r_k$$

Moreover, player 1 knows from the previous step that if he sets $z_{12} = 1$, then player 2 sets $z_{21} = 1$. His payoff from this strategy is

$$S_1(z_{-1}, z_1^2) = r_1 + \beta r_2$$

Notice that since $r_a < r_2$, we have

$$\begin{aligned} S_1(z_{-1}, z_1^a) &= r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} \sum_{k \neq 1,a} z_{ak} z_{ka} r_k < r_1 + \beta z_{a1} r_a + \beta^2 z_{a1} r_2 \sum_{k \neq 2,a} z_{ak} \\ &= r_1 + \beta z_{a1} r_a + \beta^2 \left(1 - z_{a1}\right) z_{a1} r_2 < r_1 + \beta r_2 = S_1(z_{-1}, z_1^2) \end{aligned}$$

Therefore, player 1 cannot get higher payoff by linking to someone rather than player 2. This also implies that player 1 wouldn't use mixed strategies such as $0 < z_{1i} < 1$ for some $i \in N \setminus \{1\}$. To see this, suppose player 2 invests in $a \in \Omega(z)$, i.e., $z_{1a} > 0$. Then, by the Nash property, we must have

$$S_1(z_{-1}, z_1^a) - S_1(z_{-1}, z_1^b) \geq 0$$

for all $b \in N$. However, we know that

$$S_1(z_{-1}, z_1^a) < S_1(z_{-1}, z_1^2)$$

for $a \in N \setminus \{1\}$. Thus, if player 1 moves before player 2, he sets $z_{12} = 1$.

■

(Step Six) By induction, regardless of the ordering in which players move, we will have

$$\begin{aligned} z_{i(i+1)} &= z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd \\ z_n &: free \text{ if } n : odd \end{aligned}$$

at the equilibrium of the sequential move game.

Proof. From the previous steps, we know that regardless of the ordering in which players move in the sequential move game, we will have $z_{12} = z_{21} = 1$. This is due to the fact that these players are the ones with the highest information and cannot get higher payoffs by linking to other players. Since $z_{12} = z_{21} = 1$ at the equilibrium for the sequential move game, all players will have $z_{i1} = z_{i2} = 0$ for all $i \in N \setminus \{1, 2\}$. Then, the sequential move game can be reduced to the game where we only have players $\{3, 4, ..., n\}$ with the same random ordering chosen in time zero not including players 1 and 2. In this case, player 3 and 4 will become the players with the highest level of information. Thus, from Step 1 through Step 5 we know that $z_{34} = z_{43} = 1$ at equilibrium, regardless of the ordering in which players move in the sequential move game. Then, the sequential move game can be reduced to the game where we only have players $\{5, 6, ..., n\}$ with the same random ordering chosen in time zero not including players 1, 2, 3, 4.

Continuing in the same manner, by induction, we can conclude that at the equilibrium of the sequential move game, regardless of the ordering in which players move, we have

$$z_{i(i+1)} = z_{(i+1)i} = 1 \text{ for all } i \in N \text{ where } i : odd$$

Moreover, if n is odd, then all the players except player n is matched with another player. Thus, his payoff is $S_n = r_n$, making his payoff independent of his choice variable z_n. Therefore, we have $z_n : free$ if n is odd at the subgame perfect equilibrium. ■ ■

Proposition 23 *Nash equilibrium under Cobb-Douglas link strength function may not be surplus maximizing outcome.*

Proof. Assume that there are only three players and they are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. From Proposition 22, only the Assortatively Pair Equilibrium is strongly robust. Under this equilibrium, player 1 and 2 spend all their time with each other. Therefore, player 3 is obstrasized. Assume that player 3 sets $z_{31} = 1$. We will show that a higher level of total surplus could be achieved under some conditions by changing player 1's equilibrium strategy. The following shows the partial derivative of total surplus with respect to z_{12} evaluated at $z_{12} = 1, z_{21} = 1, z_{31} = 1$.

$$\left[\frac{\partial S}{\partial z_{12}}\right]_{z_{12}=1, z_{21}=1, z_{31}=1} = -\beta\left(r_3 - r_2 + \beta r_2 + \beta r_3\right)$$

If the difference between the intrinsic values of player 2 and 3 is low enough, that is, when

$$\frac{r_2}{r_3} < \frac{1+\beta}{1-\beta}$$

the partial derivative of the total surplus with respect to z_{12} would be negative at the Assortative Pair Equilibrium with $z_{12} = z_{21} = z_{31} = 1$. Then, a higher total social surplus could be achieved by decreasing z_{12} by a small amount so that player 1 has a link with both player 1 and 2. The players' individual surpluses shows that the increase in the total surplus is a result of the increase in player 3's surplus being higher than the total decrease in player 1 and 2's surpluses.

$$\left[\frac{\partial S_1}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \beta\left(r_2 - r_3\right)$$

$$\left[\frac{\partial S_2}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = \beta\left(r_1 - \beta r_3\right)$$

$$\left[\frac{\partial S_3}{\partial z_{12}}\right]_{z_{12}=1,z_{21}=1,z_{31}=1} = -\beta\left(r_1 + \beta r_2\right)$$

■

6 Conclusion

In this paper, we analyze the formation of networks when players choose how much time to invest in other players. This is one of the few papers on weighted link formation that includes all possible paths in the network in calculation of indirect benefits. We assume that each player has an intrinsic value of information to share and one unit of endowment to invest in relationships with others. Once a direct link is formed, the information is transferred both ways with decay. Moreover, indirect links can transmit indirect information. However, the benefits from indirect information transfers are zero when two agents are connected by more than two links.

We study the model under two different link strength functions. First, we assume that the link strength is the arithmetic mean of agents' investment levels, i.e., the agents are perfect substitutes. As a positive investment of an agent is enough for a link to be formed, this specification allows players to form links unilaterally with others. We show that, when the investments are perfect substitutes, every player is connected to another either directly or indirectly with no more than two links under any Nash equilibrium. Moreover, we find that the strict Nash equilibrium structure is a star network in which all players are connected to the one with the highest total value of information.

Alternatively, we assume that the link strength function is Cobb-Douglas. Since a link between a pair of players is formed only when each of them invests in the relationship, players have to have bilateral agreement to form links with each other. Under the Cobb-Douglas link strength function, we show that paired networks, in which players are matched in pairs, are Nash equilibria. However, we also consider a sequential game in which players choose and announce their strategies publicly according to a random ordering. We show that an Assortative Pair Equilibrium, in which players are assortatively matched in pairs according to their information levels, is the only subgame perfect equilibrium of the sequential game for all possible orderings of the players. Therefore, we conclude that the Assortative Pair Equilibrium is the only strongly robust Nash equilibrium. Lastly, for both link strength functions, Nash equilibria may not be a surplus-maximizing outcome.

Equilibrium network structures vary with the link strength function. The main distinction between the different specifications of the function is the availability of the information and the element of consent to exchange information. Even though many applications have elements of both, additively separable link strength function is more appropriate for the situations when intrinsic information of an agent is available publicly; whereas, Cobb-Douglas is applicable to the situations in which both agents are required to invest in a relationship in order to exchange information. Our results are consistent with the real life applications. Particularly, pioneer agents emerge in the applications that one-sided investment is sufficient for forming a link. These situations are reminiscent to additively separable case, in which strict Nash equilibrium is a star network. Moreover, collaboration network discussed in Blau (1963) requires two-sided investments. Blau (1963) reports that the agents establish partnerships of mutual consultation and less competent agents tend to pair of as partners. This network architecture is akin to the Assortative Pair Equilibrium of Cobb-Douglas case.

Future work can proceed in a number of interesting directions. In this work, we assume that agents differ only in their intrinsic value of information. However, the agents can exhibit asymmetries in terms of endowment levels, that is, some agents may have more time to invest in relationships with others. It would be interesting to examine the effect of this additional availability on the decisions of agents. Another line of asymmetry is the coefficient for decay. In the real world, people are heterogeneous in term of their communication skills. Thus, the effectiveness of the communication may differ accordingly. This situation could be examined by allowing for different levels of decay.

Another line for future research is to weaken the assumption of the benefits from indirect information transfers are zero when two agents are connected by more than two links. Weakening this assumption increases the benefits from indirect communication. Therefore, especially in the additively separable case, we may observe different equilibrium architectures. However, we should note that weakening this assumption may result in computational difficulties.

Appendix A. Full Characterization of Three Player Game with Additively Separable Link Strength Function

A.1 Non-Cooperative Nash Equilibrium with Three Players

In this section, we fully characterize the set of Nash equilibrium under additively separable link strength with three players.

There are three players. Assume the players are ordered in terms of their information so that $r_1 > r_2 > r_3 > 0$. That is, player 1 has the most information and player 3 has the least. Each player has 1 unit of time that he can invest in relationships (links) with the other players. Let z_{ij} the amount of time player i invests in the link to j. Since player i has one unit of time to allocate, $z_{ij} + z_{ik} = 1$. The strength of the link between i and j is

$$\sigma(z_{ij}, z_{ji}) = \frac{1}{2}z_{ij} + \frac{1}{2}z_{ji}$$

Notice $\sigma(1,1) = 1$, so that $\sigma(z_{ij}, z_{ji}) \leq 1$ always holds.

The strategy set for player i consists of the time allocated to other players and is denoted z_i. The strategy set for each player must satisfy the following properties:

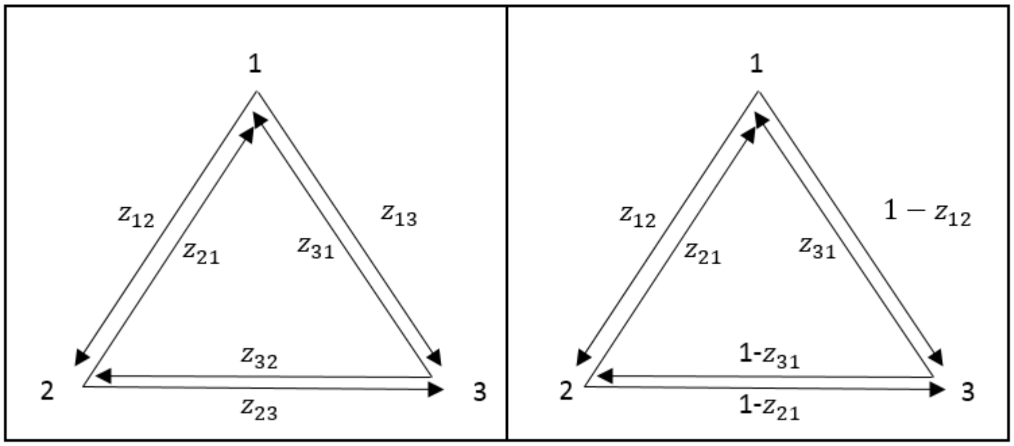
$$z_i = \{z_{ik}\}_{k \neq i}$$
$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$
$$\sum_{k \neq i} z_{ik} = 1 \text{ for each player } i$$

The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i. So, the surplus for player i can be calculated to be

$$S_i = r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{z_{kj} + z_{jk}}{2}r_j) + \beta\frac{z_{ij} + z_{ji}}{2}(r_j + \beta\frac{z_{kj} + z_{jk}}{2}r_k) \text{ for } i \neq k \neq j$$

Since there are only three players,

$$z_{ik} + z_{ij} = 1 \text{ for all } i \neq k \neq j.$$
$$z_{ij} = 1 - z_{ik}$$



We can rewrite S_i by using the figure above:

$$
\begin{aligned}
S_i &= r_i + \beta \frac{z_{ik} + z_{ki}}{2}(r_k + \beta \frac{z_{kj} + z_{jk}}{2} r_j) + \beta \frac{z_{ij} + z_{ji}}{2}(r_j + \beta \frac{z_{kj} + z_{jk}}{2} r_k) \\
&= r_i + \beta \frac{z_{ik} + z_{ki}}{2}(r_k + \beta \frac{(1 - z_{ki}) + (1 - z_{ji})}{2} r_j) + \beta \frac{(1 - z_{ik}) + z_{ji}}{2}(r_j + \beta \frac{(1 - z_{ki}) + (1 - z_{ji})}{2} r_k) \\
&= r_i + \beta \frac{z_{ik} + z_{ki}}{2}(r_k + \beta \frac{2 - z_{ki} - z_{ji}}{2} r_j) + \beta \frac{1 - z_{ik} + z_{ji}}{2}(r_j + \beta \frac{2 - z_{ki} - z_{ji}}{2} r_k)
\end{aligned}
$$

So, player i's problem is to maximize his own surplus by choosing z_{ik} subject to $0 \leq z_{ik} \leq 1$ and $z_{ij} = 1 - z_{ik}$.

Proposition 24 *The Nash equilibrium strategies for non-cooperative game for* $0 < \beta \leq 1$ *are*

$$z_{12} = 1, z_{13} = 0$$

$$z_{21} = 1, z_{23} = 0$$

$$z_{31} = 1, z_{32} = 0$$

So, the link strengths between the players are

$$\sigma_{12} = 1$$

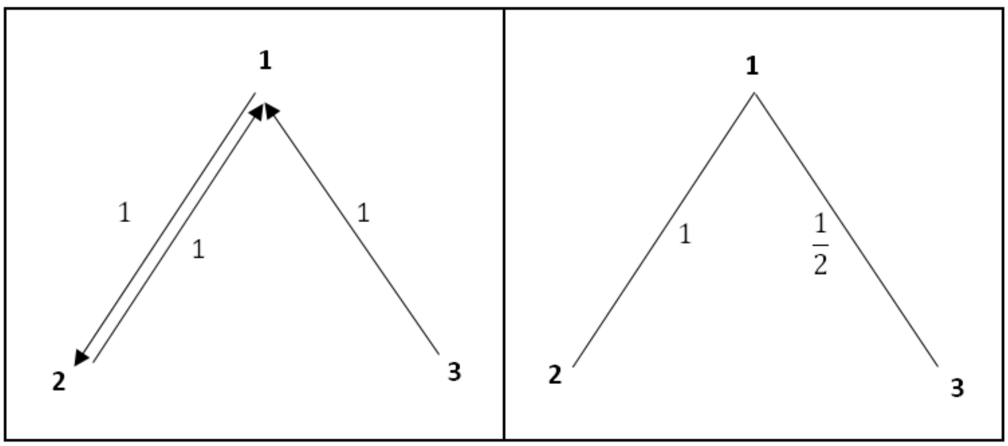
$$\sigma_{13} = \frac{1}{2}$$

$$\sigma_{23} = 0$$

and the surplus for each player is

$$\begin{aligned} S_1 &= r_1 + \beta r_2 + \frac{\beta}{2} r_3 \\ S_2 &= r_2 + \beta r_1 + \frac{\beta^2}{2} r_3 \\ S_3 &= r_3 + \frac{\beta}{2} r_1 + \frac{\beta^2}{2} r_2 \end{aligned}$$

where $S_1 > S_2 > S_3$.



Proof. If we take the derivative of player i's surplus with respect to his choice variable, we get

$$\begin{aligned} \frac{\partial S_i}{\partial z_{ik}} &= \frac{\beta}{2} r_k + \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2}) r_j - \frac{\beta}{2} r_j - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2}) r_k \\ &= \frac{\beta}{2}(r_k - r_j) - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})(r_k - r_j) \\ &= (r_k - r_j)(\frac{\beta}{2} - \frac{\beta^2}{2}(\frac{2 - z_{ki} - z_{ji}}{2})) \\ &= \frac{\beta}{2}(r_k - r_j)(1 - \beta(\frac{2 - z_{ki} - z_{ji}}{2})) \end{aligned}$$

Notice that $\frac{2 - z_{ki} - z_{ji}}{2}$ is the link strength between player k and player j. So, if $0 < \beta < 1$, then

$$1 - \beta(\frac{2 - z_{ki} - z_{ji}}{2}) > 0$$

Therefore, if $(r_k - r_j) > 0$, then $\frac{\partial S_i}{\partial z_{ik}} > 0$, making the surplus of player i increasing in z_{ik}. So, player i will invest all of his time with player k.Similarly, if $(r_k - r_j) < 0$, then $\frac{\partial S_i}{\partial z_{ik}} < 0$, making the surplus of player i decreasing in z_{ik}. So, player i will invest all of his time with player j.

Since $r_2 > r_3$, player 1 will choose $z_{12} = 1$, since $r_1 > r_3$, player 2 will choose $z_{21} = 1$ and since $r_1 > r_2$, player 3 will choose $z_{31} = 1$, making the link strengths between the players

$$\sigma_{12} = \frac{z_{12} + z_{21}}{2} = 1$$
$$\sigma_{13} = \frac{z_{13} + z_{31}}{2} = \frac{1}{2}$$
$$\sigma_{23} = \frac{z_{23} + z_{32}}{2} = 0$$

■

A.2 Surplus-Maximizing Outcome with Three Players

In this section, we fully characterize surplus-maximizing link structure under additively separable link strength with three players. We again assume the players are ordered so $r_1 > r_2 > r_3 > 0$ holds.

The surplus for player i consists of the direct and indirect information obtained by communicating with other players and is denoted as S_i and defined in the previous section as

$$S_i = r_i + \beta\frac{z_{ik} + z_{ki}}{2}(r_k + \beta\frac{z_{kj} + z_{jk}}{2}r_j) + \beta\frac{z_{ij} + z_{ji}}{2}(r_j + \beta\frac{z_{kj} + z_{jk}}{2}r_k) \text{ for } i \neq k \neq j.$$

The total surplus is calculated by

$$S = S_1 + S_2 + S_3$$

Hence, the social planner's problem is to maximize the social surplus by choosing the link strengths z_1, z_2, z_3 subject to

$$z_i = \{z_{ik}\}_{k \neq i}$$
$$0 \leq z_{ik} \leq 1 \text{ for all } k \neq i$$
$$\sum_{k \neq i} z_{ik} = 1 \text{ for each player } i$$

From the point of view of social surplus, it is only the link strengths σ_{ij} that matter, not the z_{ij}. However, the z_{ij} matter for the overall resource constraint: because $z_{ij} + z_{ik} = 1$ we obtain the following constraint on the link strengths:

$$\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$$

We claim, the planner can maximize surplus in two steps. First, choose the link strengths $\sigma_{12}, \sigma_{23}, \sigma_{13}$ to maximize surplus subject to $0 \leq \sigma_{ij} \leq 1$ and $\sigma_{12}+\sigma_{23}+\sigma_{13} = \frac{3}{2}$. Second, allocate the individual links z_{ij} so everything adds up correctly.

Claim 25 *Suppose we have σ_{ij} such that $0 \leq \sigma_{ij} \leq 1$ and $\sigma_{12}+\sigma_{23}+\sigma_{13} = \frac{3}{2}$. Then we can always find z_{ij} such that $0 \leq z_{ij} \leq 1$ and $z_{ij} + z_{ik} = 1$ and which satisfy*

$$\sigma_{12} = \frac{1}{2} z_{12} + \frac{1}{2} z_{21} \tag{6}$$

$$\sigma_{23} = \frac{1}{2} z_{23} + \frac{1}{2} z_{32} \tag{7}$$

$$\sigma_{13} = \frac{1}{2} z_{13} + \frac{1}{2} z_{31} \tag{8}$$

Proof. Without loss of generality, let σ_{23} be the weakest link and σ_{12} the strongest. Then $\sigma_{23} \leq 1/2 \leq \sigma_{12}$. Let $z_{32} = 2\sigma_{23}$ and $z_{23} = 0$, so that (7) holds. Then let $z_{31} = 1 - z_{32}$, and let z_{13} be such that (8) holds, i.e.,

$$\sigma_{13} = \frac{1}{2} z_{13} + \frac{1}{2}(1 - z_{32}) = \frac{1}{2} z_{13} + \frac{1}{2}(1 - 2\sigma_{23})$$

This means

$$z_{13} = 2(\sigma_{13} + \sigma_{23}) - 1 \tag{9}$$

Since $1/2 \leq \sigma_{12} \leq 1$ and $\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$, we have

$$\sigma_{13} + \sigma_{23} + 1 \geq \sigma_{13} + \sigma_{23} + \sigma_{12} = 3/2 \geq \sigma_{13} + \sigma_{23} + \frac{1}{2}$$

and so

$$\frac{1}{2} \leq \sigma_{13} + \sigma_{23} \leq 1$$

This means (9) lies between 0 and 1. Finally, let $z_{12} = 1 - z_{13}$ and let z_{21} be such that (6) holds, i.e.,

$$\sigma_{12} = \frac{1}{2} z_{12} + \frac{1}{2} z_{21} = \sigma_{12} = \frac{1}{2}(1 - z_{13}) + \frac{1}{2} z_{21} = \frac{1}{2}(1 - (2(\sigma_{13} + \sigma_{23}) - 1)) + \frac{1}{2} z_{21}$$

so that

$$z_{21} = 2(\sigma_{12} + \sigma_{13} + \sigma_{23}) - 2 = 1$$

We have found all the z_{ij} we are looking for.

■

From now on, then, for the sake of social surplus, we can forget the z_{ij}. The total surplus can be calculated to be

$$S = r_1 \left(1 + \beta(\sigma_{12} + \sigma_{13}) \left(1 + \beta\sigma_{23}\right)\right) + r_2 \left(1 + \beta(\sigma_{12} + \sigma_{23}) \left(1 + \beta\sigma_{13}\right)\right) + r_3 \left(1 + \beta(\sigma_{13} + \sigma_{23}) \left(1 + \beta\sigma_{12}\right)\right)$$

The social planner's problem is to maximize the social surplus by choosing the link strengths $\sigma_{12}, \sigma_{23}, \sigma_{13}$ subject to $0 \le \sigma_{ij} \le 1$ and $\sigma_{12} + \sigma_{23} + \sigma_{13} = \frac{3}{2}$. Notice that since $\sigma_{23} = \frac{3}{2} - (\sigma_{12} + \sigma_{13})$, we can rewrite the total surplus as

$$\begin{aligned} S &= r_1 \left(1 + \beta(\sigma_{12} + \sigma_{13}) \left(1 + \beta(\frac{3}{2} - \sigma_{12} - \sigma_{13})\right)\right) + r_2 \left(1 + \beta(\frac{3}{2} - \sigma_{13}) \left(1 + \beta\sigma_{13}\right)\right) \\ &\quad + r_3 \left(1 + \beta(\frac{3}{2} - \sigma_{12}) \left(1 + \beta\sigma_{12}\right)\right) \end{aligned}$$

Proposition 26 *The social planner's problem has a unique solution.*

Proof. The social planner's problem is equivalent to maximizing

$$\begin{aligned} S &= r_1 \left(1 + \beta(\sigma_{12} + \sigma_{13}) \left(1 + \beta(\frac{3}{2} - \sigma_{12} - \sigma_{13})\right)\right) + r_2 \left(1 + \beta(\frac{3}{2} - \sigma_{13}) \left(1 + \beta\sigma_{13}\right)\right) \\ &\quad + r_3 \left(1 + \beta(\frac{3}{2} - \sigma_{12}) \left(1 + \beta\sigma_{12}\right)\right) \end{aligned}$$

by choosing the link strengths σ_{12}, σ_{13} subject to $0 \le \sigma_{12} \le 1$ and $0 \le \sigma_{13} \le 1$

The Hessian matrix is given by

$$H = \begin{bmatrix} -2\beta^2 \left(r_1 + r_3\right) & -2\beta^2 r_1 \\ -2\beta^2 r_1 & -2\beta^2 \left(r_1 + r_2\right) \end{bmatrix}$$

As $-2\beta^2 \left(r_1 + r_3\right) < 0$ and $\det\left(H\right) = 4\beta^2 \left(r_1 r_2 + r_1 r_3 + r_2 r_3\right) > 0$, the Hessian matrix is negative-definite. Thus, the objective function is strictly concave in the decision variables. We know that the social planner's objective function S is continuous function over a compact set defined by $0 \le \sigma_{12} \le 1$ and $0 \le \sigma_{13} \le 1$. Thus, the solution to the problem exists. Hence, the solution is unique. ■

Note that since the objective function is strictly concave and the constraints are linear, any local maximum of the social surplus will be a global maximum. Therefore, while solving for the social planner's problem, it is sufficient to look at necessary conditions. In other words, Kuhn-Tucker conditions are sufficient for finding global maxima.

Proposition 27 *For any socially optimal solution, $\sigma_{12} > \sigma_{13} > \sigma_{23}$ holds.*

Proof. *Let* $\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*$ *be the socially optimal link structure for given* $r_1 > r_2 > r_3 > 0$ *and* $0 < \beta < 1$. *So, for all* $\epsilon > 0$

$$\begin{aligned} S(\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*) - S(\sigma_{12}^*, \sigma_{13}^* + \epsilon, \sigma_{23}^* - \epsilon) &\geqslant 0 \\ \beta\epsilon \left(r_2 - r_1)(1 - \beta\sigma_{12}^*) + \beta^2\epsilon(r_1 + r_2\right) (\sigma_{13}^* - \sigma_{23}^* - \epsilon) &\geqslant 0 \\ \beta(r_1 + r_2)(\sigma_{13}^* - \sigma_{23}^* - \epsilon) &\geqslant (r_1 - r_2)(1 - \beta\sigma_{12}^*) \\ \sigma_{13}^* - \sigma_{23}^* - \epsilon &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} \\ \sigma_{13}^* - \sigma_{23}^* &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} - \epsilon \end{aligned}$$

Since $\sigma_{12}^* \leqslant 1$ *and* $\beta < 1$, *we have* $(1 - \beta\sigma_{12}^*) > 0$. *Thus,*

$$\frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} > 0$$

since $r_1 > r_2$. *So,*

$$\begin{aligned} \sigma_{13}^* - \sigma_{23}^* - \epsilon &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} \\ \sigma_{13}^* - \sigma_{23}^* &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} - \epsilon \end{aligned}$$

Let $0 < \epsilon < \dfrac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)}$. *Then,*

$$\begin{aligned} \sigma_{13}^* - \sigma_{23}^* &\geqslant \frac{(r_1 - r_2)(1 - \beta\sigma_{12}^*)}{\beta(r_1 + r_2)} - \epsilon > 0 \\ \sigma_{13}^* &> \sigma_{23}^* \end{aligned}$$

Similarly, let $\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*$ *be the socially optimal link structure for given* $r_1 > r_2 > r_3 > 0$ *and* $0 < \beta < 1$. *So, for all* $\epsilon > 0$

$$\begin{aligned} S(\sigma_{12}^*, \sigma_{13}^*, \sigma_{23}^*) - S(\sigma_{12}^* + \epsilon, \sigma_{13}^* - \epsilon, \sigma_{23}^*) &\geqslant 0 \\ \beta\epsilon \left(r_3 - r_2)(1 - \beta\sigma_{23}^*) + \beta^2\epsilon(r_2 + r_3\right) (\sigma_{12}^* - \sigma_{13}^* - \epsilon) &\geqslant 0 \\ \beta(r_2 + r_3)(\sigma_{12}^* - \sigma_{13}^* - \epsilon) &\geqslant (r_2 - r_3)(1 - \beta\sigma_{23}^*) \\ \sigma_{12}^* - \sigma_{13}^* - \epsilon &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} \\ \sigma_{12}^* - \sigma_{13}^* &\geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon \end{aligned}$$

Since $\sigma_{23}^* \leqslant 1$ *and* $\beta < 1$, *we have* $(1 - \beta\sigma_{23}^*) > 0$. *Thus,*

$$\frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} > 0$$

since $r_1 > r_2$. *So,*

$$\sigma_{12}^* - \sigma_{13}^* - \epsilon \geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)}$$

$$\sigma_{12}^* - \sigma_{13}^* \geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon$$

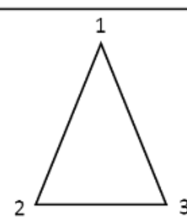
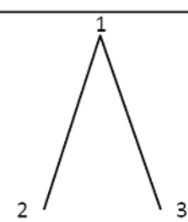
Let $0 < \epsilon < \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)}$. *Then,*

$$\sigma_{12}^* - \sigma_{13}^* \geqslant \frac{(r_2 - r_3)(1 - \beta\sigma_{23}^*)}{\beta(r_2 + r_3)} - \epsilon > 0$$

$$\sigma_{12}^* > \sigma_{13}^*$$

■

Proposition 28 *The solution to the social planner's problem is given in the following figure.*

	$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$ $(\sigma_{23} > 0)$	$\frac{r_2}{r_1} \leq \frac{2-3\beta}{2-\beta}$ $(\sigma_{23} = 0)$
$\frac{r_2}{r_3} \geq \frac{2+\beta}{2-\beta}$ $(\sigma_{12} = 1)$	$\sigma_{12} = 1$ $\sigma_{13} = \left(\frac{r_1(2-\beta) + r_2(3\beta-2)}{4\beta(r_1 + r_2)}\right)$ $\sigma_{23} = \left(\frac{r_1(3\beta-2) + r_2(2-\beta)}{4\beta(r_1 + r_2)}\right)$	$\sigma_{12} = 1$ $\sigma_{13} = \frac{1}{2}$ $\sigma_{23} = 0$
$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$ $(\sigma_{12} < 1)$	$\sigma_{12} = \left(\frac{4r_1r_2 + r_3(r_1 + r_2)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$ $\sigma_{13} = \left(\frac{4r_1r_3 + r_2(r_1 + r_3)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$ $\sigma_{23} = \left(\frac{4r_2r_3 + r_1(r_2 + r_3)(3\beta-2)}{4\beta(r_1r_2 + r_1r_3 + r_2r_3)}\right)$	$\sigma_{12} = \left(\frac{r_2(2+3\beta) - r_3(2-3\beta)}{4\beta(r_2 + r_3)}\right)$ $\sigma_{13} = \left(\frac{r_2(3\beta-2) + r_3(2+3\beta)}{4\beta(r_2 + r_3)}\right)$ $\sigma_{23} = 0$
Optimal Link Structure		

Proof. As we have pointed it out previously, Kuhn-Tucker conditions will be sufficient for global maxima.

Let us write the Lagrangian function as follows.

$$\mathcal{L} = r_1\left(1+\beta(\sigma_{12}+\sigma_{13})\left(1+\beta\sigma_{23}\right)\right)+r_2\left(1+\beta(\sigma_{12}+\sigma_{23})\left(1+\beta\sigma_{13}\right)\right)+r_3\left(1+\beta(\sigma_{13}+\sigma_{23})\left(1+\beta\sigma_{12}\right)\right)$$
$$+\lambda_1(\frac{3}{2}-\sigma_{12}-\sigma_{13}-\sigma_{23})+\lambda_2(1-\sigma_{12})+\lambda_3(1-\sigma_{13})+\lambda_4(1-\sigma_{23})$$

Thus, the solutions of the maximization of the social welfare problem is given by the following system of equations:

$$\frac{\partial\mathcal{L}}{\partial\sigma_{12}} = r_1\beta\left(1+\beta\sigma_{23}\right)+r_2\beta\left(1+\beta\sigma_{13}\right)+r_3\beta^2(\sigma_{13}+\sigma_{23})-\lambda_1-\lambda_2 \leq 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{12}}\sigma_{12} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{13}} = r_1\beta\left(1+\beta\sigma_{23}\right)+r_2\beta^2\left(\sigma_{12}+\sigma_{23}\right)+r_3\beta(1+\beta\sigma_{12})-\lambda_1-\lambda_3 \leq 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{13}}\sigma_{13} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{23}} = r_1\beta^2\left(\sigma_{12}+\sigma_{13}\right)+r_2\beta\left(1+\beta\sigma_{13}\right)+r_3\beta(1+\beta\sigma_{12})-\lambda_1-\lambda_4 \leq 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{23}}\sigma_{23} = 0$$
$$\sigma_{12}+\sigma_{23}+\sigma_{13} = \frac{3}{2}$$
$$0 \leqslant \sigma_{12} \leqslant 1$$
$$0 \leqslant \sigma_{13} \leqslant 1$$
$$0 \leqslant \sigma_{23} \leqslant 1$$
$$0 \leq \lambda_2$$
$$0 \leq \lambda_3$$
$$0 \leq \lambda_4$$

Since $r_1 > r_2 > r_3 > 0$ and by Proposition 27, the solution is characterized by four cases.

Case 1: Interior Solution: $0 < \sigma_{23} < \sigma_{13} < \sigma_{12} < 1$

The restrictions on the system of equations are:

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$
$$\frac{\partial\mathcal{L}}{\partial\sigma_{12}} = \frac{\partial\mathcal{L}}{\partial\sigma_{13}} = \frac{\partial\mathcal{L}}{\partial\sigma_{23}} = 0$$

So, the solution is

$$\sigma_{12} = \frac{4r_1r_2+r_3(r_1+r_2)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$

$$\sigma_{13} = \frac{4r_1r_3 + r_2(r_1+r_3)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$
$$\sigma_{23} = \frac{4r_2r_3 + r_1(r_2+r_3)(3\beta-2)}{4\beta(r_1r_2+r_1r_3+r_2r_3)}$$

Notice that, since $r_1 > r_2 > r_3$, we have $\sigma_{23} < \sigma_{13} < \sigma_{12}$. So, we only need to check $0 < \sigma_{23}$ and $\sigma_{12} < 1$.

For $0 < \sigma_{23}$, we need

$$\frac{r_1(r_2+r_3)}{r_2r_3} < \frac{4}{2-3\beta} \text{ if } \beta < \frac{2}{3}$$

For $\sigma_{12} < 1$, we need

$$\frac{r_3(r_1+r_2)}{r_1r_2} > \frac{4(1-\beta)}{2+\beta}$$

Now, let's assume

$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

and

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

hold. Then, we have

$$\frac{r_1}{r_2} < \frac{2-\beta}{2-3\beta}$$

and

$$\frac{r_2+r_3}{r_3} < \frac{4}{2-\beta}$$

.So for $\beta < \frac{2}{3}$ we have

$$\frac{r_1}{r_2}\frac{r_2+r_3}{r_3} < \frac{2-\beta}{2-3\beta}\frac{4}{2-\beta}$$
$$\frac{r_1(r_2+r_3)}{r_2r_3} < \frac{4}{2-3\beta}$$

Moreover, when

$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

and

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

hold, we have

$$\frac{r_1+r_2}{r_1} > \frac{4-4\beta}{2-\beta}$$

and

$$\frac{r_3}{r_2} > \frac{2-\beta}{2+\beta}$$

Then, as $0 < \beta < 1$, we obtain

$$\frac{r_1 + r_2}{r_1} \frac{r_3}{r_2} > \frac{4 - 4\beta}{2 - \beta} \frac{2 - \beta}{2 + \beta}$$
$$\frac{r_3(r_1 + r_2)}{r_1 r_2} > \frac{4(1 - \beta)}{2 + \beta}$$

Case 2: $0 < \sigma_{23} < \sigma_{13} < \sigma_{12} = 1$

The restrictions on the system of equations are:

$$\lambda_3 = \lambda_4 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = \frac{\partial \mathcal{L}}{\partial \sigma_{23}} = 0$$
$$\sigma_{23} + \sigma_{13} = \frac{1}{2}$$

So, the solution is

$$\sigma_{12} = 1$$
$$\sigma_{13} = \frac{r_1(2 - \beta) + r_2(3\beta - 2)}{4\beta(r_1 + r_2)}$$
$$\sigma_{23} = \frac{r_1(3\beta - 2) + r_2(2 - \beta)}{4\beta(r_1 + r_2)}$$

Again, notice that $\sigma_{23} < \sigma_{13} < \sigma_{12} = 1$ for $r_1 > r_2 > r_3$.

For $0 < \sigma_{23}$ and $\sigma_{13} < 1$, we need

$$\frac{r_2}{r_1} > \frac{2 - 3\beta}{2 - \beta} \text{ if } \beta < \frac{2}{3}$$

For $\lambda_2 = r_2(1 + 2\beta\sigma_{13} - \frac{3\beta}{2}) - r_3(1 + \frac{\beta}{2}) \geqslant 0$, we need

$$\frac{r_3(r_1 + r_2)}{r_1 r_2} \leqslant \frac{4(1 - \beta)}{2 + \beta}$$

Notice that

$$\frac{r_1}{r_2} < \frac{2 - \beta}{2 - 3\beta} \iff \frac{r_1}{r_1 + r_2} < \frac{2 - \beta}{4(1 - \beta)}$$

Thus, from the previous two equations, we obtain

$$\frac{r_3(r_1 + r_2)}{r_1 r_2} \leqslant \frac{4(1 - \beta)}{2 + \beta}$$
$$\frac{r_3(r_1 + r_2)}{r_1 r_2} \frac{r_1}{r_1 + r_2} \leqslant \frac{4(1 - \beta)}{2 + \beta} \frac{2 - \beta}{4(1 - \beta)}$$
$$\frac{r_3}{r_2} \leqslant \frac{2 - \beta}{2 + \beta}$$
$$\frac{r_2}{r_3} \geqslant \frac{2 + \beta}{2 - \beta}$$

Case 3: $0 = \sigma_{23}, \sigma_{13} = \frac{1}{2}, \sigma_{12} = 1$

The restrictions on the system of equations are:

$$\lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$$

So, the solution is

$$\sigma_{12} = 1$$

$$\sigma_{13} = \frac{1}{2}$$

$$\sigma_{23} = 0$$

For $\lambda_2 = r_2\beta(1 - \frac{\beta}{2}) - r_3\beta(1 + \frac{\beta}{2}) \geqslant 0$, we need

$$\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta}$$

For $\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$, we need

$$\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta}$$

Notice that

$$\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta} \iff \frac{r_2+r_3}{r_3} \geqslant \frac{4}{2-\beta}$$

From $\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta}$, $\frac{r_2+r_3}{r_3} \geqslant \frac{4}{2-\beta}$ and $\frac{r_2+r_3}{r_3} > 0$, we have

$$\frac{r_1}{r_2} \geqslant \frac{2-\beta}{2-3\beta} \iff$$
$$\frac{r_1(r_2+r_3)}{r_2r_3} \geqslant \frac{2-\beta}{2-3\beta}\frac{r_2+r_3}{r_3} \geqslant \frac{2-\beta}{2-3\beta}\frac{4}{2-\beta} = \frac{4}{2-3\beta} \text{ if } \beta < \frac{2}{3}$$
$$\frac{r_1(r_2+r_3)}{r_2r_3} > 0 > \frac{4}{2-3\beta} \text{ if } \beta > \frac{2}{3}$$

Thus, we get

$$\frac{r_1(r_2+r_3)}{r_2r_3} \geqslant \frac{4}{2-3\beta}$$

Moreover,

$$\frac{r_2}{r_1} \leqslant \frac{2-3\beta}{2-\beta} \iff \frac{r_1}{r_1+r_2} \geqslant \frac{2-\beta}{4(1-\beta)}$$

Thus, from $\frac{r_1}{r_1+r_2} \geqslant \frac{2-\beta}{4(1-\beta)}$ and $\frac{r_2}{r_3} \geqslant \frac{2+\beta}{2-\beta}$, we have

$$\frac{r_1}{r_1+r_2}\frac{r_2}{r_3} \leqslant \frac{2-\beta}{4(1-\beta)}\frac{2+\beta}{2-\beta}$$
$$\frac{r_1 r_2}{r_3(r_1+r_2)} \leqslant \frac{2+\beta}{4(1-\beta)}$$
$$\frac{r_3(r_1+r_2)}{r_1 r_2} \leqslant \frac{4(1-\beta)}{2+\beta}$$

Case 4: $0 = \sigma_{23}, 0 < \sigma_{13} < \sigma_{12} < 1$

The restrictions on the system of equations are:

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_{12}} = \frac{\partial \mathcal{L}}{\partial \sigma_{13}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$$

So, the solution is

$$\sigma_{12} = \frac{r_2(2+3\beta) - r_3(2-3\beta)}{4\beta(r_2+r_3)}$$
$$\sigma_{13} = \frac{r_2(3\beta-2) + r_3(2+3\beta)}{4\beta(r_2+r_3)}$$
$$\sigma_{23} = 0$$

For $\sigma_{13} = \frac{3}{2} - \sigma_{12}$ and $\sigma_{12} < 1$, we have $\frac{1}{2} < \sigma_{13}$.Thus, we need

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$

For $\frac{\partial \mathcal{L}}{\partial \sigma_{23}} \leqslant 0$, we need

$$\frac{r_1(r_2+r_3)}{r_2 r_3} \geqslant \frac{4}{2-3\beta}$$

Note that we have

$$\frac{r_2}{r_3} < \frac{2+\beta}{2-\beta}$$
$$\frac{r_2+r_3}{r_3} < \frac{4}{2-\beta}$$

Then,

$$\frac{4}{2-3\beta} \leqslant \frac{r_1(r_2+r_3)}{r_2 r_3} \leqslant \frac{r_1}{r_2}\frac{(r_2+r_3)}{r_3} < \frac{r_1}{r_2}\frac{4}{2-\beta}$$
$$\frac{4}{2-3\beta} < \frac{r_1}{r_2}\frac{4}{2-\beta}$$
$$\frac{r_2}{r_1} > \frac{2-3\beta}{2-\beta}$$

■

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